The Delay-Limited Capacity Region of OFDM Broadcast Channels*

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Abstract

In this work, the delay limited capacity (DLC) of orthogonal frequency division multiplexing (OFDM) systems is investigated. The analysis is organized into two parts. In the first part, the impact of system parameters on the OFDM DLC is analyzed in a general setting. The main results are that under weak assumptions the maximum achievable single user DLC is almost independent of the distribution of the path attenuations in the low signal-to-noise (SNR) region but depends strongly on the delay spread. In the high SNR region the roles are exchanged. Here, the impact of delay spread is negligible while the impact of the distribution becomes dominant. The relevant asymptotic quantities are derived without employing simplifying assumptions on the OFDM correlation structure. Moreover, for both cases it is shown that the DLC is maximized if the total channel energy is uniformly spread, i.e. the power delay profile is uniform. It is worth pointing out that since universal bounds are obtained the results can also be used for other classes of parallel channels with block fading characteristic. The second part extends the setting to the broadcast channel and studies the corresponding OFDM DLC BC region. An algorithm for computing the OFDM BC DLC region is presented. To derive simple but smart resource allocation strategies, the principle of rate water-filling employing order statistics is introduced. This yields analytical lower bounds on the OFDM DLC region based on orthogonal frequency division multiple access (OFDMA) and ordinal channel state information (CSI). Finally, the schemes are compared to an algorithm using full CSI.

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Index Terms

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I. Introduction

Multiple degrees of freedom in fading channels allow reliable communication in each fading state under a long term power constraint. This is due to the possibility of recovering the information from several independently faded copies of the transmitted signal. The rate achievable in each fading state is called *zero outage capacity* or alternatively *delay limited capacity* (DLC) [1]. Not only multiple input multiple output (MIMO) channels but also frequency selective multipath channels offer multiple degrees of freedom. This is in contrast to single antenna Rayleigh flat fading channels, where a DLC exists only if zero is not in the support of the fading distribution [2]. Since the DLC does not involve a decoding delay over multiple fading blocks if the variation of the fading process is slow enough, it can be considered as an appropriate limit for delay sensitive services, which become more and more important recently. Unlike the DLC, the traditional ergodic capacity strongly depends on the correlation structure of the fading process and generally implies an infinite decoding delay.

This work investigates the DLC of frequency selective multipath channels in the context of an orthogonal frequency division multiplexing (OFDM) broadcast (BC) channel. OFDM can be considered as a special case of parallel fading channels with correlated fading process. Pioneering work on this topic was carried out in [3] where the general single user outage capacity was investigated of which the DLC is a special case. The optimal power control law is derived which is such that bad channels below some threshold are simply switched off. In [4] the limiting performance in the high signal-to-noise (SNR) regime of the DLC of parallel fading channels with multiple antennas was characterized assuming that the fading distribution is continuous. In [5] the impact of spatial correlations was studied. Further work was carried out in [6], [7]. Unfortunately, these results do not carry over to the OFDM case: since the subcarriers are highly correlated due to oversampling of the channel in the frequency domain the fading distribution is commonly degenerated which significantly complicates the analysis. This particularly affects the critical impact of the delay spread and the number of subcarriers. Hence, the behavior of the OFDM DLC is not clear yet.

Our main contributions are the following: First, we analyze the impact of system parameters such as delay spread, power delay profile (or the multipath intensity profile) and the fading distribution for the single user DLC in a general setting. We focus on two cases: the behaviour at high SNR and at low SNR. Such approach has been frequently used in the analysis of channel capacity even if the capacity itself is not completely known [8], [9]. The low SNR regime is characterized by its first and second order expansion. It is shown that to become first order optimal it is sufficient to serve only one of the best subcarriers regardless of the fading distribution. The corresponding limit is almost independent of the fading distribution. The second order limit is also calculated and depends generally on the number of supported subcarriers, i.e. when the fading distribution contains point masses. The quantities are shown to exist for a large class of fading distributions and are explicitly calculated in terms of the delay spread for the Rayleigh fading case. For the high SNR regime several universal bounds are calculated. These bounds culminate in a convergence theorem that generally characterizes the high SNR behaviour under very weak assumptions. Most important, there will be no need for concepts like almost sure convergence of the empirical distributions etc. (as used in [4]) and one approaches ergodic capacity relatively fast (the difference decreases with order $\log^{-1}(L)$ in a channel with L uniform taps and Rayleigh fading) even if the fading gains are *not independent*. The corresponding convergence processes are characterized. It is worth pointing out that these results not only hold for the OFDM case but also for other classes of parallel block fading channels. Finally, we provide a convergence result with respect to ergodic capacity.

In the second part we focus on the broadcast scenario, complicating the analysis significantly. To begin, we present an algorithm which is capable of evaluating the OFDM BC DLC region up to any finite accuracy. This is a challenging problem, since for each fading state the minimum sum power supporting a set of rates has to be found [10]. To get a guideline for algorithm design, we subsequently derive lower bounds on the single user OFDM DLC based on rate water-filling and order statistics. These single user bounds are the point of origin for the development of simple analytical lower bounds on the OFDM DLC region based on orthogonal frequency division multiple access (OFDMA). The involved use of order statistics has a positive impact on the feedback protocol. In the low SNR regime, nearly the entire OFDM BC DLC region and in the high SNR regime a significant part of it can be achieved with these schemes without any form of time-sharing. Further, a practical OFDMA algorithm based on rate water-filling assuming perfect

CSI is introduced. This scheme outperforms the bounds based on partial CSI and might serve as a benchmark for other OFDMA minimum sum power algorithms. The results are illustrated by simulations.

The remainder of this paper is organized as follows: Section II introduces the OFDM system model. In Section III and IV, the behavior at low and high SNR is studied in detail for the single user case. Section V contains the characterization and computation of the OFDM broadcast channel DLC region. Subsequently, in Section VI lower bounds on the OFDM BC DLC region are derived. We conclude with some final remarks final Section VII.

A. Notation

All terms will be arranged in boldface vectors (where m refers to users, k to subcarriers, l to path delays as a guideline) and the corresponding indices will be omitted if there is no ambiguity. Common vector norms (such as $\|\cdot\|_1$ for the l_1-norm) will be employed. The expression $z \sim \mathcal{CN}(0,1)$ means that the random variable $z = x + jy$ is complex Gaussian distributed, i.e. the real and imaginary parts are independently Gaussian distributed with zero mean and variance 1/2: $x, y \sim \mathcal{N}(0, 1/2)$. The expectation operator (e.g. with respect to the fading process) will be denoted as $\mathbb{E}$ (respectively $\mathbb{E}_{\mathbf{h}}$). $\Pr(A)$ denotes the probability of an event A. We write $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ if $x \to 0$ (or $x \to \infty$) and all logarithms are to the base e unless explicitely defined in a different manner.

II. System Model

Assume an OFDM broadcast channel with M users from the set $\mathcal{M} := \{1, \ldots, M\}$ and K subcarriers from the set $\mathcal{K} := \{1, \ldots, K\}$. The sampled frequency response of each user is by means of Fast Fourier Transform (FFT) given by

$$\tilde{h}_{m,k} = \sum_{l=1}^{L} \tilde{c}_{m,l}\, e^{-\frac{2\pi j (l-1)(k-1)}{K}}, \quad k \in \mathcal{K} \tag{1}$$

where $L \leq K$ is the delay spread and $\tilde{c}_{m,l}$ are the complex path gains which are i.i.d. according to $\tilde{c}_{m,l} \sim \mathcal{CN}(0, \sigma_{m,l})$. The vector $\boldsymbol{\sigma}_m = [\sigma_{m,1}, ..., \sigma_{m,K}]^T$ is called the power delay profile (PDP) of the mth user's channel. The variances $\sigma_{m,l}$ are assumed to be strictly positive for all m and l and the channel energy is normalized $\|\boldsymbol{\sigma}_m\|_1 = 1$ for all users $m \in \mathcal{M}$. We say that the channel of user m has a uniform PDP if $\sigma_{1,m} = \ldots = \sigma_{L,m}$ and a non-uniform PDP otherwise. Note that

in practice the PDP is typically non-uniform. Furthermore, the channel gains are not spread over the entire frequency band. Then, our results hold approximately and serve also as a performance limit. The channel (path) gains are defined as $h_{m,k} := |\tilde{h}_{m,k}|^2$ (respectively $c_{m,l} := |\tilde{c}_{m,l}|^2$). The distribution of the channel gains is called the (joint) fading distribution. In case of complex Gaussian distributed path gains, i.e. $\tilde{c}_k \sim \mathcal{CN}\left(0, 1/L\right)$ the channel gains follow a exponential distribution with $\Pr\left(h_k > x\right) = e^{-x}$. This case corresponds to Rayleigh fading.

Let $x_{m,k}$ with power $p_{m,k}$ be the signal transmitted from the base station to user m on carrier k and let $\mathbf{x}_m = [x_{m,1}, ..., x_{m,K}]^T, \mathbf{p}_m = [p_{m,1}, ..., p_{m,K}]^T$ the stacked vector of transmit signals and transmit powers for user m, respectively. Further assume that the system is limited by a sum power constraint $\mathbb{E}\left(\sum_{m=1}^{M} \|\mathbf{x}_m\|_2^2\right) \leq KP^*$ where P^* is the power budget. Then the system equation on each subcarrier k can be written as

$$y_{m,k} = \tilde{h}_{m,k} \sum_{i \in \mathcal{M}} x_{i,k} + n_{m,k}, \quad k \in \mathcal{K},$$

where $y_{m,k}$ is the signal received by user m on subcarrier k, and $n_{m,k}$ represents (without loss of generality) normalized noise $n_{m,k} \sim \mathcal{CN}\left(0, 1\right)$. Let us define a decoding order π, such that user $\pi\left(M\right)$ is decoded first, followed by user $\pi\left(M-1\right)$ and so on. Assuming ideal superposition coding at the transmitter and successive interference cancellation with the decoding order π at the receivers, the rate of user $\pi(m)$ is then given by

$$\tilde{R}_{\pi(m)} = \frac{1}{K} \sum_{k=1}^{K} \log\left(1 + \frac{h_{\pi(m),k} p_{\pi(m),k}}{1 + h_{\pi(m),k} \sum_{n<m} p_{\pi(n),k}}\right) \quad \text{in [nats/Hz/s].} \tag{2}$$

For an instantaneous channel realization $\mathbf{h} = [h_{1,1}, ..., h_{1,K}, h_{2,1}, ..., h_{M,K}]^T$ (respectively $\mathbf{c} = [c_{1,1}, ..., c_{1,L}, c_{2,1}, ..., c_{M,L}]^T$ is the vector of path gains) the OFDM broadcast channel capacity region is given by the union over all power allocations fulfilling the sum power constraint P^* and over the set of all possible permutations Π:

$$\mathcal{C}_{\mathrm{BC}}\left(\mathbf{h}, P^*\right) \equiv \bigcup_{\substack{\pi \in \Pi \\ \sum_{m=1}^{M} \|\mathbf{p}_m\|_1 \leq P^*}} \left\{\mathbf{R} : R_{\pi(m)} \leq \tilde{R}_{\pi(m)}\ , m \in \mathcal{M}\right\}$$

Here, $\mathbf{R} = [R_1, ..., R_M]^T$ and $\tilde{R}_{\pi(m)}$ is the rate of user $\pi\left(m\right)$ defined in (2).

III. OFDM SINGLE USER DLC

A. An implicit formulation of the single user OFDM DLC

First let us define the single user DLC C_d for OFDM, which is a special case of parallel fading channels. The user index m is omitted in this section.

Definition 1: A rate R^* is achievable with limited delay (zero outage) under a long term power constraint P^* if and only if for any fading state $\mathbf{h}$ there exists a power allocation $\mathbf{p}'$ solving

$$\min ||\mathbf{p}'||_1 \quad \text{subj. to} \quad R^* \leq R(\mathbf{h}, \mathbf{p}') \tag{3}$$

and

$$\mathbb{E}_{\mathbf{h}} \left(||\mathbf{p}'||_1 \right) \leq P^*.$$

Furthermore R^* is called the *delay limited capacity* $C_d(P^*)$ if and only if

$$\mathbb{E}_{\mathbf{h}} \left(||\mathbf{p}'||_1 \right) = P^*.$$

Obviously, the delay-limited capacity is smaller than the ergodic capacity under the same average power constraint because the requested rate has to be supported in every fading state. Observe that we choose to state the DLC region as a definition rather than a theorem as done in [1]. There it is shown that our definition coincides with rates that guarantee arbitrary small erroneous decoding probability (dependent on the coding delay) for any jointly stationary, ergodic fading process when the codewords can be chosen as a function of the realization of the fading process. This can be interpreted as a performance limit for traffic where the information has to be delivered within one coding frame for almost all fading realizations and the channel varies slowly enough such that the transmitter can track the channel. Moreover, by our definition we see that since the sum power minimization problem has to be solved in every fading step the DLC represents also a performance limit for practical minimum sum power algorithms. We will present an example in Sec. VI.

The single user DLC was already examined by several authors in the context of systems with multiple antennas and parallel independent fading channels [4], [5]. Nevertheless, we re-derive the DLC here using the principle of *rate water-filling*, which leads to an interesting perspective. This characterization of the DLC will be used later on to derive lower bounds on the OFDM broadcast channel DLC region in Section VI. Denoting the rate on subcarrier k as r_k, the

optimization problem for each fading state in (3) is equivalent to

$$\min \sum_{k=1}^{K} p_k - \lambda \sum_{k=1}^{K} r_k \quad \text{subj. to} \quad \sum_{k=1}^{K} r_k \geq K C_d.$$

Using the relation between power and rate on each subcarrier this can be expressed as

$$\min \sum_{k=1}^{K} \left(\frac{e^{r_k} - 1}{h_k} - \lambda r_k \right),$$

where $\lambda > 0$ is to be chosen so that $\sum_{k=1}^{K} r_k = K C_d$. The resulting optimality conditions are given by

$$\left[\frac{e^{r_k}}{h_k} - \lambda \right]^- = 0 \quad \forall k, \quad \sum_{k=1}^{K} r_k = K C_d \tag{4}$$

with $[\cdot]^- = \min\{\cdot, 0\}$. Note that by taking logarithms and solving for r_k eqn. (4) can be interpreted as a water-filling solution with respect to the rates r_k. Combining the optimality conditions for all K subcarriers and solving for the Lagrangian multiplier λ yields

$$\lambda = \frac{\exp\left(\frac{C_d K}{d_{\mathbf{h}}}\right)}{\prod_{k \in \mathcal{D}(\mathbf{h})} h_k^{d_{\mathbf{h}}^{-1}}}, \tag{5}$$

where the random variable $\mathcal{D}(\mathbf{h}) \subseteq \mathcal{K}$ denotes the set of *active* subcarriers and $d_{\mathbf{h}} := |\mathcal{D}(\mathbf{h})|$. Note that the allocated power is given by $p_k = \lambda - h_k^{-1}$ for any $k \in \mathcal{D}(\mathbf{h})$ and zero otherwise. Substituting (5) in the average power expression given by

$$P^* = \mathbb{E}_{\mathbf{h}} \left(\sum_{k=1}^{K} p_k(\lambda) \right) = \mathbb{E}_{\mathbf{h}} \left(\sum_{k=1}^{K} \left[\lambda - \frac{1}{h_k} \right]^+ \right) \tag{6}$$

with $[\cdot]^+ = \max\{\cdot, 0\}$ we obtain the single user OFDM delay limited capacity C_d with power constraint P^*:

$$P^* = \mathbb{E}_{\mathbf{h}} \left(\frac{d_{\mathbf{h}} \exp\left(\frac{C_d K}{d_{\mathbf{h}}}\right)}{K \prod_{k \in \mathcal{D}(\mathbf{h})} h_k^{d_{\mathbf{h}}^{-1}}} \right) - \frac{1}{K} \mathbb{E}_{\mathbf{h}} \left(\sum_{k \in \mathcal{D}(\mathbf{h})} \frac{1}{h_k} \right) \tag{7}$$

Since the denominator in (7) can be bounded by a constant, the delay limited capacity C_d is greater than zero if and only if

$$\int_{\mathbb{R}_+^K} \frac{1}{\prod_{k \in \mathcal{D}(\mathbf{h})} h_k^{d_{\mathbf{h}}^{-1}}} \, dF_{\mathbf{h}}(\mathbf{h}) < \infty. \tag{8}$$

Here, $F_{\mathbf{h}}$ denotes the joint fading distribution function. The class of fading distributions for which (8) holds is called *regular* in [4]. It will become apparent in the following that the correlation structure of the channel gains in OFDM provides the main challenge in proving and analyzing regularity according to (8).

B. *Suboptimal power allocation strategies*

Let us introduce a suboptimal power allocation for the single-user case that is used in VI. It is evident from the expression for the single user DLC that the major difficulty is the rate water-filling operation for each channel realization. To circumvent this difficulty which results in a prohibitive complexity for the multi-user case we introduce *the notion of rate water-filling* for average channel realizations. The idea is to use simply the information which subcarrier is the best, the second best and so on and to allocate fixed rate budgets to the in that way ordered subcarriers. For the analysis we need the following definitions. For a given vector $\mathbf{h}$ of real elements let us introduce the total ordering

$$h_{k[K]} \geq h_{k[K-1]} \geq \ldots \geq h_{k[1]},$$

i.e. $h_{k[1]}$ is the minimum value and $h_{k[M]}$ is the maximum value. If $\mathbf{h}$ is a random variable then the distribution of $h_{k[p]}$ is known to be the *p-th order statistics*. The the p-th order statistics can be explicitely given for K independent random variables h_k with distribution F and density f from standard books. The p-th order density is given by

$$f_{h_{k[p]}}(x) = Kf(x)\binom{K-1}{p}F^{p-1}(x)(1-F(x))^{K-p} \tag{9}$$

Based on the order information and the distribution we can now deduce a fixed rate allocation on the subcarriers, avoiding the water-filling procedure in each fading state. The idea is to allocate a fixed rate budget to the p-th ordered subcarrier. Defining the terms

$$\zeta_p = \int_0^\infty h^{-1}\, dF_{h_{k[p]}}(h)$$

and using these factors the optimal rate allocation is now given by solving the optimization problem

$$\min_{R_{k[p]}\geq 0} \frac{e^{R_{k[p]}}-1}{\zeta_p^{-1}} - \lambda R_k, \quad \sum_{k=1}^{K} R_{k[p]} = KC_d,$$

which we have already solved by rate water-filling given by

$$\left[\frac{\log(e)\, e^{R_{k[p]}}}{\zeta_p^{-1}} - \lambda\right]^{-} = 0, \quad \sum_{p=1}^{K} R_{k[p]} = KC_d.$$

Obviously, this suboptimal scheme has some impact on the *feedback protocol*: Since only the order statistics are exploited, it seems to suffice to feed back the ordering of the subcarriers. This affords much less feedback capacity than perfect channel knowledge would require. However,

it is not straightforward to translate rate water-filling to a power allocation strategy, since for achieving a certain rate the channel has to be perfectly known. Allocating fixed power budgets is possible. The consequence is, that since the p-th order channel is a random variable, mutual information becomes a random variable once again not guaranteeing a certain rate in each state. However, the variance becomes much smaller.

There is an interesting second power allocation where the powers asserted to the subcarriers are all the same. It is easy to see that the allocation according to

$$p_k = \frac{e^R}{\mathbb{E}_{\mathbf{h}}\left(\prod_{k=1}^K h_k^{-1/K}\right)}, \quad k \in \mathcal{K}, \tag{10}$$

always leads to a rate higher than the requested rate with equality in the high SNR region. Hence, this is also a suboptimal solution. The bounds are illustrated in Fig. 1.

It is of great interest to understand the impact of the delay spread L and the power delay profile $\boldsymbol{\sigma}$ as well as the fading distribution on the OFDM delay limited capacity. In case of the OFDM broadcast channel, an analytical characterization is nearly impossible. Thus we carry out an analysis for the single user OFDM DLC in the following. Note that this matches the behavior on the axes of the OFDM BC DLC region where only one user is active. Hence the results give insights for the broadcast case as well. Since the expression in (7) is still very complicated, we focus on the behavior in the low and the high SNR regime and carry out a detailed analysis.

IV. Impact of system parameters

A. Scaling in low SNR

1) Impact of delay spread and fading distribution: First, we characterize the first and second order behavior at low SNR. For ease of notation we define $h_\infty := \|\mathbf{h}\|_\infty$.

Proposition 1: Suppose that $\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right) < \infty$. Then, the following limit holds:

$$\lim_{P^* \to 0} \frac{C_d\left(P^*\right)}{P^*} = \frac{1}{\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right)} \tag{11}$$

Proof: Our starting point is (7) where we use the McLaurin-expansion of the exponential function $\exp\left(x\right) = 1 + x + o\left(x\right)$ up to the linear term to obtain a lower bound on the required power P^*.

Fix now $\epsilon > 0$ and use the following strategy: set the 'virtual' channel gain of any subcarrier k with $h_k \geq h_\infty - \epsilon$ to h_∞. Denote by $\chi_{\mathbf{h}}\left(\epsilon\right)$ the multiplicity of the number of subchannels that

are assigned the maximum channel gain by this strategy for any channel realization. Using (7) we have for sufficiently small P^* an upper bound on C_d which is given by

$$P^* \geq \mathbb{E}_{\mathbf{h}}\left(\frac{\chi_{\mathbf{h}}(\epsilon) + C_d K}{K h_\infty}\right) - \frac{1}{K}\mathbb{E}_{\mathbf{h}}\left(\frac{\chi_{\mathbf{h}}(\epsilon)}{h_\infty}\right) \tag{12}$$

$$= C_d \mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_\infty}\right) \tag{13}$$

and we obtain

$$C_d \leq \frac{P^*}{\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right)} \tag{14}$$

for any $\epsilon > 0$.

For the upper bound on P^* fix $\epsilon' > 0$. This time set the number of supported subcarriers to one and support one of the set with maximum channel gain. Then we have for sufficiently small P^*

$$P^* \leq \mathbb{E}_{\mathbf{h}}\left(\frac{1 + (1+\epsilon') C_d K}{K h_\infty}\right) - \frac{1}{K}\mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_\infty}\right) \tag{15}$$

Hence we have

$$C_d \geq \frac{P^*}{(1+\epsilon')\,\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right)} \tag{16}$$

for any $\epsilon' > 0$. Combining both the lower and upper bound yields the desired result. ■

This quantity also reveals the minimum energy per bit, at which reliable communication is possible under a limited delay. This is in analogy to [8], where this quantity was derived for the ergodic capacity of a Gaussian channel. Moreover, the lemma states that albeit it is generally suboptimal, serving one of the best subcarriers becomes optimal in the low SNR region. This can be easily seen since the multiplicity $\chi_{\mathbf{h}}(\epsilon)$ of the attained maximum channel gain vanishes in the expressions for the lower and upper bound.

To gain more insights, the remaining sub-linear term defined by

$$\Delta_d(P^*) := C_d'(0)\,P^* - C_d(P^*) \tag{17}$$

is calculated next. The following proposition tells us that while for the linear term it did not matter if the distribution contains point masses it does matter for the sub-linear term:

Proposition 2: The following limit for the sub-linear term holds:

$$\lim_{P^*\to 0} \frac{\Delta_d(P^*)}{(P^*)^2} = \frac{K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1} h_\infty^{-1}\right)}{2\mathbb{E}_{\mathbf{h}}^3\left(h_\infty^{-1}\right)} \tag{18}$$

Here, $\chi_{\mathbf{h}}$ is the (random) multiplicity of subchannels with maximum channel gain.

Proof: Our starting point is again (7) where we now use the McLaurin-expansion of the exponential function up to the quadratic term, i.e. $\exp(x) = 1 + x + 0.5x^2 + o(x^2)$.

By the same strategy as above fix $\epsilon > 0$ and set the "virtual" channel gain of any subcarrier k with $h_k \geq h_\infty - \epsilon$ to h_∞. Then, we obtain for sufficiently small P^*

$$P^* \geq \mathbb{E}_{\mathbf{h}} \left(\frac{C_d + 0.5C_d^2 K \chi_{\mathbf{h}}^{-1}(\epsilon)}{h_\infty} \right). \tag{19}$$

The equation is an upward open parabola in C_d where one zero is negative and one is positive where the latter is increasing in P^*. Solving this equation for C_d yields the inequality

$$\begin{aligned} C_d(P^*) \leq & -\frac{\mathbb{E}_{\mathbf{h}}(h_\infty^{-1})}{K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1}\right)} \\ & + \sqrt{\frac{\mathbb{E}_{\mathbf{h}}^2(h_\infty^{-1})}{K^2\mathbb{E}_{\mathbf{h}}^2\left(\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1}\right)} + \frac{2P^*}{K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1}\right)}}. \end{aligned} \tag{20}$$

Expanding the square root function yields

$$C_d(P^*) \leq \frac{1}{\mathbb{E}_{\mathbf{h}}(h_\infty^{-1})} P^* - \frac{K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1}\right)}{2(1+\epsilon)\mathbb{E}_{\mathbf{h}}^3(h_\infty^{-1})} (P^*)^2. \tag{21}$$

Subtracting the first order expression (11) from (21) we arrive for some $\epsilon' > 0$ at

$$\begin{aligned} \Delta_d(P^*) &\geq \frac{K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1}\right)}{2(1+\epsilon)\mathbb{E}_{\mathbf{h}}^3(h_\infty^{-1})} (P^*)^2 \\ &\geq \frac{K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1} h_\infty^{-1}\right) - \epsilon'}{2(1+\epsilon)\mathbb{E}_{\mathbf{h}}^3(h_\infty^{-1})} (P^*)^2 \end{aligned} \tag{22}$$

and thus have established a lower bound on $\Delta_d(P^*)$ for any $\epsilon, \epsilon' > 0$. The last inequality (22) follows from the following argument (which is frequently used in the sequel): observe that $\chi_{\mathbf{h}}(\epsilon) \geq 1$ and, almost surely with respect to the fading distribution, for any realization $\mathbf{h}$ we have

$$\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1} \to \chi_{\mathbf{h}}^{-1} h_\infty^{-1}, \quad \epsilon \to 0, \tag{23}$$

and provided that $\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1} h_\infty^{-1}\right) \leq \mathbb{E}_{\mathbf{h}}(h_\infty^{-1}) < \infty$ we obtain by dominated convergence [11]:

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1}(\epsilon) h_\infty^{-1}\right) = \mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1} h_\infty^{-1}\right) \tag{24}$$

In analogy to the derivation of the first order behavior we get

$$\Delta_d(P^*) \leq \frac{(1+\epsilon') K\mathbb{E}_{\mathbf{h}}\left(\chi_{\mathbf{h}}^{-1} h_\infty^{-1}\right)}{2\mathbb{E}_{\mathbf{h}}^3(h_\infty^{-1})} (P^*)^2. \tag{25}$$

Combining (22) and (25) leads to the desired result. ■

Corollary 1: Since $\chi_{\mathbf{h}}(\epsilon) \geq 1$ a simple upper bound on the second order term in (18) is given by

$$\limsup_{P^* \to 0} \frac{\Delta_d\left(P^*\right)}{\left(P^*\right)^2} \leq \frac{K}{2\left[\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right)\right]^2}. \tag{26}$$

Note that the bound from Corollary 1 is consistent with the result in [12] where it is shown that

$$C_d\left(P^*\right) \sim \frac{1}{K} \log\left(1 + \frac{KP^*}{\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right)}\right), \quad P^* \to 0. \tag{27}$$

This can be easily checked by differentiating the expression in (27) twice.

Proposition 3: Suppose that the joint fading distribution is absolute continuous. Then, the following limit holds:

$$\lim_{P^* \to 0} \frac{\Delta_d\left(P^*\right)}{\left(P^*\right)^2} = \frac{K}{\left[\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right)\right]^2} \tag{28}$$

Proof: We have to prove that the set of events where the maximum is taken on by more than one subcarrier has probability zero.

By the absolute continuity of the joint fading distribution (which is preserved under unitary mappings) this is equivalent to show that the set that contains all events where the maximum is not unique has Lebesgue measure zero. To see this we consider sets of the form

$$\left\{\mathbf{h} \in \mathbb{R}_+^K : \bigcup_{1 \leq i \leq K} \left\{h_{\pi(1)} = \ldots = h_{\pi(i)}, h_{\pi(i+1)}, \ldots, h_{\pi(K)}\right\}\right\}$$

where π is the permutation that yields an decreasing order among the channel gains

$$h_{\pi(1)} \geq h_{\pi(2)} \geq \ldots \geq h_{\pi(K)}$$

Observe that any set has measure zero. Since there can only be 2^K such possible sets the union of these sets has measure zero as well. Now, as our regarded set is in the union of the constructed set, the set has also measure zero. ∎

Hence, appealing to Prop. 1, 2 and 3 the forthcoming analysis reduces to the study of the expected maximum of the channel gains. However, the expressions do not show how the DLC depends on the system parameters which we investigate by means of an asymptotic analysis, i.e. for large L, K. This analysis turns out to be quite accurate even for very small L.

Remark 1: It is important to note that for the asymptotic analysis we will let go L and K, $K \geq L$, to infinity which is indicated by the index n, i.e. $K_n, L_n \to \infty$ as $n \to \infty$. We assume that for all L_n the complex path gain vectors $\tilde{\mathbf{c}}_n = [\hat{c}_1, \ldots, \hat{c}_{L_n}]^T$ are defined on the same probability space, i.e. to each $\tilde{\mathbf{c}}_n \in \mathbb{C}^{L_n}$ there is by means of (1) $\tilde{\mathbf{h}}_n = [\hat{h}_1, \ldots, \hat{h}_K]^T \in \mathbb{C}^{K_n}$.

While the distribution does not change for the L_{n-1} first random variables of the vector $\tilde{\mathbf{c}}_n$ the distribution of the K_{n-1} first random variables in $\tilde{\mathbf{h}}_n$ might change since they depend on the FFT structure. We have to keep this in mind for the forthcoming analysis.

It was shown in [13] that the (continuous) maximum value of the frequency response equals $\log(L)$ with large probability even for moderate L for the following distributions:

$$\begin{aligned}&\tilde{c}_i \text{ independent and iid (both in real and imaginary parts) } \forall i,\\ &\mathbb{E}_{\mathbf{h}}\left(\Re e^2\left(\tilde{c}_1\right)\right) = .5/L, \mathbb{E}_{\mathbf{h}}\left(e^{j\omega\Re e(\tilde{c}_1)}\right) = e^{-.25L^{-1}\omega^2+\sum_{l=3}^{5} a_l\omega^l + O\left(\omega^6\right)},\\ &\text{for all } |\omega| \leq d, \quad d>0, a_3, a_4, a_5 \in \mathbb{C}\end{aligned} \tag{29}$$

Here, $\Re e\,(c)$ denotes the real part of the complex number c ($\Im m$ is the imaginary part). Note that the condition on the characteristic function of the real part of the path gains implies finiteness of the moments up to order six of the corresponding distribution. The following theorem proves that the $\log(L)$ result holds also for the maximum of the sampled frequency response regardless of the sampling set. Furthermore, it reveals the exceptional role of uniform PDP.

Theorem 1: Suppose that the fading distribution belongs to $\mathcal{F}_L$ then we have

$$\Pr\left(\log(L) - g(L) \leq h_\infty \leq \log(L) + g(L)\right) = 1 - O\left(\log^{-4}(L)\right)$$

with $g(L) = 4\log\left[\log(L)\right]$ for large L and arbitrary $K \geq L$. Furthermore, the upper bound

$$\Pr\left(h_\infty \geq \log(L) + g(L)\right) \leq O\left(\log^{-4}(L)\right)$$

also holds when the PDP is non-uniform.

Proof: see Appendix A. ■

We can apply this result to the DLC assuming uniform PDP where we have to show that from the convergence in probability given in Theorem 1 follows convergence of the expected maximum of the channel gains. This can be achieved if the set of distributions is somewhat more restricted compared to (29) in the sense that the their behavior in the neighborhood of the zero is ”sufficiently well”. By this we mean, that the distribution function of c_1 is Lipschitz continuous in an ϵ-neighborhood of the zero. Let us denote this class of fading distributions by:

$$\begin{aligned}\mathcal{F}_{\text{Lo}}(k_s) := \{&\text{the conditions (29) on } \tilde{c}_i \text{ hold } \forall i\\ &\mathbf{h} \in \mathbb{R}_+^K \text{ is generated from } \tilde{\mathbf{c}} \in \mathbb{C}_+^L \text{ by means of (1)}\\ &F_{c_1}(x) \leq k_s x, 0 \leq x < \epsilon, k_s, \epsilon \in \mathbb{R}_{++}\}\end{aligned}$$

The Lipschitz continuity is essential in the next Lemma.

Lemma 1: Suppose that the fading distribution belongs to $\mathcal{F}_{\text{Lo}}\left(k_s\right)$ then the following limit holds for sufficiently large L and arbitrary $K \geq L$:

$$\lim_{P^* \to 0} \frac{C_d\left(P^*\right)}{P^*} = \log\left(L\right) + O\left(\log\left[\log\left(L\right)\right]\right)$$

Furthermore, the DLC is maximized (with respect to the leading order term) by uniform PDP ("order-optimal").

Proof: The last statement follows from Theorem 1 and, hence, we assume uniform PDP. By the same theorem there exist constants $\gamma, \kappa \in \mathbb{R}_{++}$ so that

$$\begin{aligned} &\Pr\left(\log\left(L\right) - \gamma\log\left[\log\left(L\right)\right] \leq h_\infty \leq \log\left(L\right) + \gamma\log\left[\log\left(L\right)\right]\right) \\ &\geq 1 - \frac{\kappa}{\log^\gamma\left(L\right)} \end{aligned}$$

for sufficiently large L. Setting $\epsilon_- = 1 - \epsilon$ and $\epsilon_+ = 1 + \epsilon$ where $\epsilon := \gamma\log\left[\log\left(L\right)\right]/\log\left(L\right)$ the expectation can be written as

$$\begin{aligned} &\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right) \\ &= \mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\middle| h_\infty \in \left[\epsilon_-\log\left(L\right), \epsilon_+\log\left(L\right)\right]\right)\,\Pr\left(h_\infty \in \left[\epsilon_-\log\left(L\right), \epsilon_+\log\left(L\right)\right]\right) \\ &+ \mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\middle| h_\infty < \epsilon_-\log\left(L\right)\right)\,\Pr\left(h_\infty < \epsilon_-\log\left(L\right)\right) \\ &+ \mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\middle| h_\infty > \epsilon_+\log\left(L\right)\right)\,\Pr\left(h_\infty > \epsilon_+\log\left(L\right)\right). \end{aligned}$$

Note that the crucial part is to derive an upper bound on the conditional expectation in the second term on the RHS of the last equation, i.e. when the maximum h_∞ is small. We will show now that the conditional expectation is bounded too. Proceeding with the standard inequality $\mathbb{E}\left(|X|\right) \leq 1 + \sum_{n=1}^{\infty}\Pr\left(|X| \geq n\right)$ and using the "trick" that we can apply it to the conditional probability measure as well, the second term is given by

$$\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\middle| h_\infty < \epsilon_-\log\left(L\right)\right) \leq 1 + \sum_{i=1}^{+\infty}\frac{\Pr\left(\left\{h_\infty \leq \frac{1}{i}\right\} \cap \left\{h_\infty < \epsilon_-\log\left(L\right)\right\}\right)}{\Pr\left(\left\{h_\infty < \epsilon_-\log\left(L\right)\right\}\right)}.$$

Using the inequality $\left\|\mathbf{c}\right\|_1 = \left\|\mathbf{h}\right\|_1/K \leq h_\infty$, and $\left\|\mathbf{c}\right\|_\infty \leq \left\|\mathbf{c}\right\|_1$, define $P\left(x\right) := \Pr\left(\left\{h_\infty < x\right\}\right)$ and we obtain by independence of the path gains

$$\begin{aligned} &\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\middle| h_\infty < \epsilon_-\log\left(L\right)\right) \\ &\leq 1 + \sum_{i=1}^{1+\lfloor k_s\rfloor}\frac{P\left(\frac{1}{i}\right)}{P\left(\epsilon_-\log\left(L\right)\right)} + \frac{1}{P\left(\epsilon_-\log\left(L\right)\right)}\sum_{i=2+\lfloor k_s\rfloor}^{+\infty}\Pr\left(\bigcap_{l=1}^{L}\left\{c_l \leq \frac{1}{i}\right\}\right) \end{aligned} \tag{30}$$

where $\lfloor k_s \rfloor$ is the greatest natural number below k_s. Hence

$$\begin{aligned}
\text{RHS of (30)} &= 2 + \lfloor k_s \rfloor + \frac{1}{P\left(\epsilon_- \log\left(L\right)\right)} \sum_{i=2+\lfloor k_s \rfloor}^{+\infty} F_{c_1}^L \left(\frac{1}{i}\right) \\
&\leq 2 + \lfloor k_s \rfloor + \frac{k_s^L}{P\left(\epsilon_- \log\left(L\right)\right)} \sum_{i=2+\lfloor k_s \rfloor}^{+\infty} \frac{1}{i^L}. \qquad (31)
\end{aligned}$$

In the second step we assumed Lipschitz continuity of the path gain distribution function F_{c_1} with Lipschitz constant below or equal k_s, i.e. $F_{c_1}\left(x\right) \leq k_s x$. The series in (31) can be upper bounded as follows:

$$\begin{aligned}
\sum_{i=2+\lfloor k_s \rfloor}^{+\infty} \frac{1}{i^L} &\leq \sum_{i=2+\lfloor k_s \rfloor}^{+\infty} \int_{i-1}^{i} \frac{1}{x^L}\, dx \\
&= \int_{1+\lfloor k_s \rfloor}^{+\infty} \frac{1}{x^L}\, dx \\
&= \frac{1}{-L+1} \left[x^{-L+1}\right]_{1+\lfloor k_s \rfloor}^{\infty} \\
&\leq \frac{1}{L-1} k_s^{-L+1}
\end{aligned}$$

Hence, we obtain finally

$$\text{RHS of (31)} \leq 2 + \lfloor k_s \rfloor + \frac{1}{P\left(\epsilon_- \log\left(L\right)\right)} \frac{k_s}{L-1}.$$

The last term is finite for $L > 1$. Hence, we have the upper bound

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right) &\leq \epsilon_-^{-1} \log^{-1}\left(L\right) + \frac{\left(2 + \lfloor k_s \rfloor\right)\kappa}{\log^{\gamma}\left(L\right)} + \frac{k_s}{L-1} + \epsilon_+^{-1} \log^{-1}\left(L\right) \frac{\kappa}{\log^{\gamma}\left(L\right)} \\
&= \epsilon_-^{-1} \log^{-1}\left(L\right) \left(1 + \frac{\left(2 + \lfloor k_s \rfloor\right)\epsilon_- \kappa}{\log^{\gamma-1}\left(K\right)} + \frac{k_s \epsilon_- \log\left(L\right)}{L-1} + \frac{\epsilon_- \kappa}{\epsilon_+ \log^{\gamma}\left(L\right)}\right) \\
&= \epsilon_-^{-1} \log^{-1}\left(L\right) \left(1 + O\left(\frac{1}{\log^{\gamma-1}\left(L\right)}\right)\right).
\end{aligned}$$

for $\gamma > 1$. Therefore, we have finally

$$\begin{aligned}
C_d\left(P^*\right) &\geq \frac{1}{K} \log\left(1 + KP^* \epsilon_- \log\left(L\right) \left[1 + O\left(\frac{1}{\log^{\gamma-1}\left(L\right)}\right)\right]^{-1}\right) \\
&= \frac{1}{K} \log\left(1 + KP^* \epsilon_- \log\left(L\right) \left[1 - O\left(\frac{1}{\log^{\gamma-1}\left(L\right)}\right)\right]\right).
\end{aligned}$$

A lower bound is obviously

$$\mathbb{E}_{\mathbf{h}}\left(h_\infty^{-1}\right) \geq \epsilon_+^{-1} \log^{-1}\left(L\right) \left(1 - \frac{\kappa}{\log^{\gamma}\left(L\right)}\right)$$

and we have

$$C_d(P^*) \leq \frac{1}{K}\log\left(1 + KP^*\epsilon_+ \log(L)\left(1 - \frac{\kappa}{\log^{\gamma}(L)}\right)^{-1}\right)$$
$$= \frac{1}{K}\log\left(1 + KP^*\epsilon_+ \log(L)\left[1 + O\left(\frac{1}{\log^{\gamma-1}(L)}\right)\right]\right)$$

and the result follows. ■

We can conclude from the proof as follows:

Corollary 2: The DLC is finite if there are either

- at least two independent channel gains in the frequency domain or
- at least two independent path gains in the time domain

with Lipschitz continuous marginal distribution function in the neighborhood of the zero.

Interestingly, the DLC compares favorably by the factor $\log(L)$ with the capacity of AWGN in the low SNR regime. Hence, the delay spread governs the DLC in this region.

Let us make the bound explicit for the Rayleigh fading case. Note that due to the sum of independent complex path gains the samples of the frequency response are approximately complex Gaussian distributed anyway (however, the exact distribution would be very difficult to evaluate) [14].

Lemma 2: Suppose L divides K and under the assumption of complex Gaussian distributed path gains with uniform PDP the maximum channel gain is enclosed by the inequalities

$$\Pr\left(\log(L) - \gamma\log[\log(L)] \leq h_\infty \leq \log(L) + \gamma\log[\log(L)]\right)$$
$$\geq 1 - \frac{\kappa}{\log^{\gamma}(L)}$$

where $\gamma > 0$ and $\kappa \geq \frac{K}{L - \log^{-\gamma}(L)}$ and γ, L sufficiently large such that the bound makes sense.

Proof: see Appendix B. ■

We can apply the result again to the DLC.

Corollary 3: Under the assumption of complex Gaussian distributed path gains with uniform PDP the low SNR DLC is enclosed by

$$\frac{1}{K}\log\left(1 + \kappa_1 KP^* \log(L)\right) \leq C_d(P^*) \leq \frac{1}{K}\log\left(1 + \kappa_2 KP^* \log(L)\right)$$

where

$$\kappa_1 := \left(1 - \frac{\gamma\log[\log(L)]}{\log(L)}\right)\left(1 - \frac{1.1\kappa}{\log^{\gamma-1}(L)}\right)$$

and

$$\kappa_2 := \left(1 + \frac{\gamma \log\left[\log\left(L\right)\right]}{\log\left(L\right)}\right)\left(1 + \frac{1.1\kappa}{\log^{\gamma}\left(L\right)}\right)$$

for any $\gamma > 1, \kappa > 0,$ and $L > 1$ from Lemma 2.

Proof: The result follows if the constants in Lemma 2 are used in the proof of Lemma 1 and by simple numerical check. ■

Remark 2: If L does not divide K it seems very difficult to get results that are asymptotically tight. However, good bounds are easily obtained by observing that the frequency response cannot arbitrarily overshoot between the samples. In fact, for the upper bound we can use a grid $K' = aL$, say by a factor $a > 1$, and by collecting these samples in $\mathbf{h}^{(a)}$ we have [15]

$$\|\mathbf{h}\|_\infty \leq \left\|\mathbf{h}^{(a)}\right\|_\infty \cos^{-1}\left(\frac{\pi}{2a}\right).$$

For the lower bound we set $a = 1$ and have

$$\|\mathbf{h}\|_\infty \geq \left\|\mathbf{h}^{(1)}\right\|_\infty \cos\left(\frac{\pi L}{2K}\right).$$

Since we have just derived bounds for $\left\|\mathbf{h}^{(a)}\right\|_\infty, a \geq 1$, we can tackle also the general case.

An illustration is shown in Fig. 2 where we calculate (27) for different but low SNR. It is observed that the approximations are quite accurate for small L. The behavior of the OFDM DLC at low SNR and the corresponding first and second order approximations are depicted in Figures 4 and 5. It can be seen that the region where the approximations hold diminishes as the number of degrees of freedom increases. The bounds can be used to roughly estimate the performance of e.g. a cellular system at the cell border. Even though we have made not effort to optimize the bound we found by simulations that for Rayleigh fading with small delay spread the error is within reasonable span of the optimal curves. However, it is worth noting that for large ratios of L and K the range where the bound makes sense becomes increasingly small.

2) Impact of power delay profile: The impact of the PDP has been touched already in Lemma 1 proving the "order-optimality" of uniform PDP. Let us now investigate the general case. The following expression can be used to get a bound for arbitrary PDP.

Proposition 4: For sufficiently low SNR an upper bound on the DLC is given by:

$$C_d\left(P^*\right) \leq \frac{1}{K}\log\left(1 + \frac{K^{(1+\alpha)}P^*}{\mathbb{E}\left(\|\mathbf{c}\|_1^{-1}\right)}\right) \tag{32}$$

Here, $\alpha \in (0,1)$ is a global parameter that can be numerically optimized. The lower bound is independent of the order of the elements of the PDP and concave, thus *Schur-concave*.

Proof: The proof follows from Prop. (4) and the inequality chain $\|\mathbf{h}\|_1 \geq \|\mathbf{h}\|_\infty \geq K^{-1}\|\mathbf{h}\|_1 = \|\mathbf{c}\|_1$. Since both lower and upper bound are tight only in identifiable special cases there is some $\alpha \in (0,1)$ that can be numerically found. The Schur-concavity is obtained from Prop. B.2. in [16, pp. 287] since $\|\mathbf{c}\|_1^{-1}$ is symmetric and convex and the expectation $\mathbb{E}\left(\|\mathbf{c}\|_1^{-1}\right)$ is independent of the ordering of the PDP. ■

The Schur-concavity implies that for fixed L and normalized PDP the bound is larger if the elements of the PDP vector are more "spread out". If L is large the bound approaches the low SNR AWGN capacity time a factor K^α for uniform PDP due to the strong law of large number, i.e. $\|\mathbf{c}\|_1 \to 1$ almost surely, and hence α is of order $\log\left[\log(L)\right]/\log(L)$.

B. Scaling in high SNR

Defining the quantity $\overline{h} := \prod_{k=1}^K h_k^{-1/K}$ it was proved in [4] that by using the suboptimal power control law (10) $C_d(P^*) \geq \log\left(P^*/\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)\right)$ provided that $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty$, i.e. for regular fading distributions. We can extend this result to an upper bound without using any simplifying assumptions on the fading distribution.

Proposition 5: Suppose that $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty$. For sufficiently large P^* the DLC is upper bounded by

$$C_d(P^*) \leq \log\left(\frac{P^*\left(1+\frac{1}{K}\right)}{\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)}\right).$$

Proof: We can use the following strategy for an upper bound C_d: fix $\epsilon > 0$ and suppose that for any fading state $\mathbf{h}$ we set the values that are below ϵ to ϵ. In other words we do not allow "virtual" channel gains below ϵ.

Define $h_k^\epsilon := \max\{h_k, \epsilon\}$. Then, using (7) we have for sufficiently large P^*

$$\begin{aligned} P^* \geq & \mathbb{E}_{\mathbf{h}}\left(e^{C_d}\prod_{k=1}^K (h_k^\epsilon)^{-\frac{1}{K}}\right) - \frac{1}{K}\sum_{k=1}^K \mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_k^\epsilon}\right) \\ = & \mathbb{E}_{\mathbf{h}}\left(e^{C_d}\prod_{k=1}^K (h_k^\epsilon)^{-\frac{1}{K}}\right) - \mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_1^\epsilon}\right). \end{aligned} \tag{33}$$

Obviously, the second term grows without bound as $\epsilon \to 0$ for many fading distributions such as Rayleigh fading. Furthermore the growth depends on P^*. Let us bound this term as follows:

we have

$$\mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_1^{\epsilon}}\right) = \int_0^{\infty} \frac{1}{h_1^{\epsilon}} dF_{\mathbf{h}}\left(h_1\right)$$
$$\leq \epsilon^{-1}.$$

Clearly, the term ϵ is related to P^*. Since the maximum channel gain is at least ϵ and the underlying optimal power control law is water-filling the above equation (33) is certainly true if

$$P^* \geq \frac{K}{\epsilon}$$

which is a rough estimation. Hence, we obtain

$$\mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_1^{\epsilon}}\right) \leq \frac{P^*}{K}$$

and finally for any $\epsilon > 0$

$$C_d \leq \log\left(\frac{P^*\left(1+\frac{1}{K}\right)}{\mathbb{E}_{\mathbf{h}}\left(\prod_{k=1}^{K}\left(h_k^{\epsilon}\right)^{-\frac{1}{K}}\right)}\right).$$

Now observe that $\prod_{k=1}^{K}\left(h_k^{\epsilon}\right)^{-\frac{1}{K}} \leq \overline{h}$ and

$$\lim_{\epsilon \downarrow 0} \prod_{k=1}^{K}\left(h_k^{\epsilon}\right)^{-\frac{1}{K}} = \overline{h}.$$

Hence, by dominated convergence

$$\mathbb{E}_{\mathbf{h}}\left(\prod_{k=1}^{K}\left(h_k^{\epsilon}\right)^{-\frac{1}{K}}\right) \to \mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)$$

provided that

$$\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty.$$

∎

The proposition states that as long as $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty$ the DLC lies in some target corridor determined by $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)$. The following proposition was proved in [4] where it is shown that, under appropriate circumstances, serving all subcarriers equally is sufficient to achieve the limiting performance.

Proposition 6: Suppose that $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty$. If the joint distribution is continuous, then

$$C_d\left(P^*\right) \sim \log\left(\frac{P^*}{\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)}\right).$$

Hence, appealing to Prop. 5, and 6 it suffices to evaluate the term $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)$ in the high SNR regime. The problem of whether or not the high SNR quantity is non-zero is not touched upon in [4]. Let us therefore derive general conditions under which this is true. As before we could assume that the fading distribution is Lipschitz continuous on $\mathbb{R}_{+}^{K}$ which is still too restricting though.

Remark 3: Note that one is attempted to derive these conditions from the easier low SNR quantity since the DLC increases with SNR. However, this is in general misleading since the joint fading distribution might e.g. contain point masses on the boundary of the positive orthant rendering the high SNR term infinite while the low SNR term still provides a proper lower bound.

The following proposition puts a general lower bound on the DLC and will be used in Lemma (3). In order to not overload the formalism we assume that the joint distribution on a equidistant subset with some distance $a \in \mathbb{N}$ of $b \in \mathbb{N}$ subcarriers possesses a density as follows. Define $\mathbf{h}\left(a, k_1\right) := \left[h_{k_1+0a}, h_{k_1+a} \ldots, h_{k_1+ab}\right]^T, k_1 \in (1, \ldots, a)$. We define the following class:

$$
\begin{aligned}
\mathcal{F}_{\text{Hi}}\left(c_s\right) := \Big\{ & F_{\mathbf{h}(a,k_1)} \text{ has density } f_{\mathbf{h}(a,k_1)}; \\
& f_{\mathbf{h}(a,k_1)} \text{ and all marginal densities are bounded by } c_s \\
& \text{within } 1/b\text{-neighborhood for all } k_1 \in (1, \ldots, a) \\
& \mathbf{h} \in \mathbb{R}_{+}^{K} \text{ is generated from } \tilde{\mathbf{c}} \in \mathbb{C}_{+}^{L} \text{ by means of (1)} \Big\}
\end{aligned}
$$

Observe that we have not excluded point masses in this definition.

Proposition 7: Suppose that the joint fading distribution on this subset belongs to $\mathcal{F}_{\text{Hi}}\left(c_s\right)$. Then, for sufficiently large P^* the following lower bound holds:

$$
C_d\left(P^*\right) \geq \log\left(P^*\right) - \log\left(b\, c_s \left(\frac{b}{b-1}\right)^b + \left(1 - c_s\right) b\right) \tag{34}
$$

If a can be chosen to be $K/2$, i.e. $b = 2$ for even K we have:

$$
C_d\left(P^*\right) \geq \log\left(P^*\right) - \log\left(6 c_s + 2\right) \tag{35}
$$

Proof: see Appendix C. ∎

Clearly, the bound can be improved in case of independent subcarriers which will be used in Cor. (2).

Proposition 8: Suppose that the joint fading distribution can be written as $F_{\mathbf{h}(a,k_1)} \equiv \prod_{i=1}^{b} f$ for all k_1[1], i.e. subsets of the subcarriers are independent, and that the marginal channel gain density f is finite everywhere. Then, for sufficiently large P^* the lower bounds (34),(35) can be improved to give:

$$C_d\left(P^*\right) \geq \log\left(P^*\right) - b\log\left(\frac{b}{b-1}\right)$$
$$- \log\left[\left(\sum_{i\geq 1} f\left(a_i^-\right) - f\left(b_i^-\right)\right)^{1/b} - \sum_{i\geq 1} a_i^{\frac{b-1}{b}}\left(f\left(b_i^+\right) - f\left(a_i^+\right)\right)\right]$$

Here, $0 \leq a_i^{-/+} < b_i^{-/+} \leq \infty$ are interval boundaries of $\mathrm{supp}\left(f\right)$ such that $f'\left(h\right) \leq 0, h \in \left[a_i^-, b_i^-\right]$ and $f'\left(h\right) \geq 0, h \in \left[a_i^+, b_i^+\right]$. If b is large and $f'\left(h\right) \leq 0, h \in \mathbb{R}$, with $f\left(0\right) = 1$ (e.g. Rayleigh fading), then $\left(b/b-1\right)^b \rightarrow e^1$ and hence $C_d\left(P^*\right) \geq \log\left(P^*\right) - 1$.

Proof: see Appendix D. ■

For the next proposition we need explicit the properties of the FFT.

Proposition 9: Suppose that L, K are even and that the densities of real and imaginary parts of the complex path gain distribution fulfill $f_{\Re e(\tilde{c}_i),\Im m(\tilde{c}_i)}\left(x\right) \leq \sqrt{v_i}e^{-\alpha|x|^2}$. Then, the following lower bound holds:

$$C_d\left(P^*\right) \geq \log\left(P^*\right) - \log\left(\frac{4\pi^L \prod_{k=1}^{L} v_k^{1/2}}{L^L}\left(\frac{L}{\alpha}\right)^{L-1}\right)$$

Proof: see Appendix E. ■

The latter proposition is universal but tailored to the Rayleigh fading case. More sophisticated bounding techniques can be obtained from mixing the techniques of Prop. (7) to Prop. (9) as discussed in the remark of Appendix E. In the following we assume without loss of generality that the marginal fading distributions are such that $\mathbb{E}_{\mathbf{h}}\left(\log\left(h_k\right)\right)$ is independent of $k \in \mathcal{K}$ (and finite) where K is supposed here to be even.

Lemma 3: Suppose that $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty$. For sufficiently large P^* the DLC is upper bounded by

$$C_d\left(P^*\right) \leq \log\left(P^*\right) + H\left(F_{\mathbf{h}}\right) + \frac{1}{K}$$

[1]It is straightforward to see (by the structure of the FFT) that if the fading distribution is generated by a complex path gain distribution of which the density can be written as $f_{\tilde{c}_i}\left(\tilde{c}_i\right), \tilde{c}_i \in \mathbb{C}$, where $f_{\tilde{c}_i}$ is invariant under complex rotations, then the fading distribution is also invariant regarding k_1.

where

$$H\left(F_{\mathbf{h}}\right) := \int_0^\infty \log\left(h\right)\, dF_{h_1}\left(h\right)$$

and F_{h_1} is the marginal fading distribution. Furthermore, suppose that there is a sequence of fading distributions in $\mathcal{F}_{\mathrm{Hi}}\left(c_s\right)$ such that $K^{-1}\sum_{k=1}^{K}\log\left(h_k\right) \to H\left(F_{\mathbf{h}}\right)$ in probability. Then, the bound is asymptotically tight, i.e.

$$C_d\left(P^*\right) \to \log\left(P^*\right) + H\left(F_{\mathbf{h}}\right).$$

Proof: Setting

$$\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) = \mathbb{E}_{\mathbf{h}}\left(\exp\left[\log\left(\overline{h}\right)\right]\right)$$

we get by Jensen's inequality

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}}\left(\exp\left[\log\left(\overline{h}\right)\right]\right) \geq \exp\left(\mathbb{E}\left[\log\left(\overline{h}\right)\right]\right)\\ &= \exp\left(\int_0^\infty \log\left(h\right)\, dF_{h_1}\left(h\right)\right)\end{aligned}$$

which is already the desired upper bound provided that $H\left(F_{\mathbf{h}}\right) < \infty$ or equivalently $\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) < \infty$.

In order to show the tightness of the upper bound define the following random variable (i.e. partial sums):

$$h^{(K)} := -\frac{1}{K}\sum_{k=1}^{K}\log\left(h_k\right). \tag{36}$$

Suppose that $h^{(K)} \to H\left(F_{\mathbf{h}}\right)$ in probability. We have to show that

$$\begin{aligned}&E_{\mathbf{h}}\left(\exp\left[h^{(K)}\right]\right)\\ &\to \exp\left(-\int_0^\infty \log\left(h\right)\, dF_{h_1}\left(h\right)\right), K \to \infty,\end{aligned}$$

which would follow if $h^{(K)}$ is uniformly bounded in K but is not true for the situation at hand. Using the set function $\mathbb{I}\left\{\cdot\right\}$ and writing for some $C > 0$

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\right) = \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\mathbb{I}\left\{h^{(K)} \leq C\right\}\right)\\ &+ \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\mathbb{I}\left\{h^{(K)} > C\right\}\right)\\ &\leq \mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right), \exp\left(C\right)\right\}\right)\\ &+ \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\mathbb{I}\left\{h^{(K)} > C\right\}\right)\end{aligned}$$

yields for the first term on the RHS

$$\begin{aligned}
&\mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right),\exp\left(C\right)\right\}\right)\\
&\to \exp\left[\int_0^\infty \log\left(h\right)\, dF_{h_1}\left(h\right)\right], \quad K\to\infty,
\end{aligned}$$

since $h^{(K)} \to H\left(F_{\mathbf{h}}\right)$ in probability and $\min\left\{\exp\left(h^{(K)}\right), C\right\}$ is uniformly bounded (and C sufficiently large!). Hence, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\right)\\
&\leq H\left(F_{\mathbf{h}}\right) + \limsup_{n\to\infty} \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\mathbb{I}\left\{h^{(K)} > C\right\}\right)\\
&= H\left(F_{\mathbf{h}}\right) + \limsup_{n\to\infty} \int_{\mathbb{R}_+^K} \overline{h}\,\mathbb{I}\left\{h^{(K)} > C\right\}\, dF_{\mathbf{h}}\left(\mathbf{h}\right)
\end{aligned}$$

Fix $\epsilon > 0$ we have by the inequality shown in (48, Appendix C)

$$\int_{\mathbb{R}_+^K} \overline{h}\,\mathbb{I}\left\{h^{(K)} > C\right\}\, dF_{\mathbf{h}}\left(\mathbf{h}\right) \leq \prod_{l=1}^{K/2} \int_{\mathbb{R}_+^K\cap\Omega_C} \overline{h}_l^{\frac{1}{2}}\overline{h}_{l+K/2}^{\frac{1}{2}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right)$$

where $\Omega_C := \left\{\mathbf{h}\in\mathbb{R}_+^K : h^{(K)} > C\right\}$. Then by geometric mean inequality and a "sandwich" argument

$$\begin{aligned}
&\int_{\mathbb{R}_+^K} \overline{h}\,\mathbb{I}\left\{h^{(K)} > C\right\}\, dF_{\mathbf{h}}\left(\mathbf{h}\right)\\
&\leq \frac{2}{K}\sum_{l=1}^{K/2}\left(\int_{\mathbb{R}_+^K\cap\Omega_C}\left(\overline{h}_l^{\frac{1}{2}}\overline{h}_{l+K/2}^{\frac{1}{2}} - \left(\overline{h}_l^{\epsilon}\right)^{\frac{1}{2}}\left(\overline{h}_{l+K/2}^{\epsilon}\right)^{\frac{1}{2}}\right)\, dF_{\mathbf{h}}\left(\mathbf{h}\right)\right)\\
&+ \frac{2}{K}\sum_{l=1}^{K/2}\left(\int_{\mathbb{R}_+^K\cap\Omega_C}\left(\overline{h}_l^{\epsilon}\right)^{\frac{1}{2}}\left(\overline{h}_{l+K/2}^{\epsilon}\right)^{\frac{1}{2}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right)\right)\\
&\leq \epsilon^{-1}\Pr\left(h^{(K)} > C\right) + C_\epsilon
\end{aligned}$$

where $\Pr\left(h^{(K)} > C\right) \to 0$ for $K\to\infty$ (C again sufficiently large). The remaining constant $C_\epsilon > 0$ can be made arbitrarily small independent of K since $F_{\mathbf{h}} \in \mathcal{F}_{\mathrm{Hi}}\left(c_s\right)$ uniformly. On the other hand, since

$$\liminf_{n\to\infty} \mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right),\exp\left(C\right)\right\}\right) \geq H\left(F_{\mathbf{h}}\right)$$

we have the desired result. ■

The required convergence in probability follows if:

- either, the subcarriers are independent (or a subset),

- or the logarithmic channel gains are uncorrelated (or a subset) with

$$\frac{1}{K^2}\sum_{k=1}^{K}\mathbb{E}_{\mathbf{h}}\left(\log^2\left(h_k\right)\right)\to 0,\quad K\to\infty. \tag{37}$$

Interestingly, since $H\left(F_{\mathbf{h}}\right) < 0$ there is always a loss in capacity compared to AWGN. Furthermore, observe that the second statement (37) is substantially weaker than independence. It suggests that ergodic capacity can be achieved even if the subcarriers are not independent which is discussed in the next subsection. We can apply this result to the Rayleigh fading case.

Theorem 2: Under the assumption of complex Gaussian distributed path gains, the upper bound

$$C_d\left(P^*\right)\leq\log\left(P^*\right)+H\left(F_{\mathbf{h}}\right)+\frac{1}{K}$$

holds where

$$H\left(F_{\mathbf{h}}\right):=\int_0^\infty\log\left(h\right)\exp\left(h\right)dh\approx -0.58.$$

The bound is asymptotically tight for uniform PDP and sequences $\left(L_n,K_n\right)$ where L_n divides K_n, i.e.

$$C_d\left(P^*\right)\to\log\left(P^*\right)+H\left(F_{\mathbf{h}}\right),\quad n\to\infty.$$

The convergence speed is given by:

$$C_d\left(P^*\right)\geq\log\left(P^*\right)+H\left(F_{\mathbf{h}}\right)+O\left(\log^{-1}\left(K\right)\right)$$

(see eqn. (38) for constants)

Proof: The first part of the theorem follows immediately from Lemma 3 and the fact that subsets of L subcarriers are independent. It remains to provide an explicit expression for the convergence speed. Let $h^{(K)}$ be defined as as in (36). Then by Lemma 3 we have to investigate the following terms

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\right)\\&\leq\mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right),C\right\}\right)+\mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\mathbb{I}\left\{h^{(K)}>C\right\}\right)\end{aligned}$$

where $C > H(F_{\mathbf{h}})$. Defining now the event $\mathcal{A} = \left\{ \left| h^{(K)} - \exp\left(H\left(F_{\mathbf{h}}\right)\right) \right| \leq \epsilon \right\}$ and its complement $\mathcal{A}^C$ we have

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right)\right) \leq\ & \mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right), C\right\} \mathbb{I}\left\{\mathcal{A}\right\}\right) \\
& + \mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right), C\right\} \mathbb{I}\left\{\mathcal{A}^C\right\}\right) \\
& + \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right) \mathbb{I}\left\{h^{(K)} > C\right\}\right)
\end{aligned}$$

The first two terms can be bounded as follows: since

$$\mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right), C\right\} \mathbb{I}\left\{\mathcal{A}\right\}\right) \leq \exp\left(H\left(F_{\mathbf{h}}\right)\right) + \epsilon$$

and

$$\mathbb{E}_{\mathbf{h}}\left(\min\left\{\exp\left(h^{(K)}\right), C\right\} \mathbb{I}\left\{\mathcal{A}^C\right\}\right) \leq C \Pr\left(\mathcal{A}^C\right)$$

we need a bound on the probability $\Pr\left(\mathcal{A}^C\right)$ dependent on ϵ. The probability can be easily upper bounded by Tschebyscheff's inequality, i.e.

$$\Pr\left(\mathcal{A}^C\right) \leq \frac{\sigma^2}{K\epsilon^2}$$

where

$$\sigma^2 = \mathbb{E}_{\mathbf{h}}\left(\left[\log\left(h_1\right) - \mathbb{E}_{\mathbf{h}}\left(\log\left(h_1\right)\right)\right]^2\right)$$

and by choosing some sufficiently slowly converging zero sequence, e.g. $\epsilon_K = O\left(1/\log\left(K\right)\right)$.

The third term can be upper bounded by observing that:

$$\begin{aligned}
& \mathbb{E}_{\mathbf{h}}\left(\exp\left(h^{(K)}\right) \mathbb{I}\left\{h^{(K)} > C\right\}\right) \\
& = \int_{\mathbb{R}_+^K} \overline{h}\, \mathbb{I}\left\{h^{(K)} > C\right\} \, dF_{\mathbf{h}}\left(\mathbf{h}\right) \\
& \leq \left(\int_{\mathbb{R}_+^K} \left(\overline{h}\right)^2 \, dF_{\mathbf{h}}\left(\mathbf{h}\right)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+^K} \left(\mathbb{I}\left\{h^{(K)} > C\right\}\right)^2 \, dF_{\mathbf{h}}\left(\mathbf{h}\right)\right)^{\frac{1}{2}} \\
& \leq \sqrt{\exp\left(1\right)} \left(\Pr\left(\left\{h^{(K)} > C\right\}\right)\right)^{\frac{1}{2}}
\end{aligned}$$

In the last inequality we employed Prop. 8 for the first integral. The probability can again be tackled with Tschebyscheff's inequality. It follows

$$\Pr\left(\left\{h^{(K)} > C\right\}\right) \leq \frac{\sigma^2}{K\left(C - H\left(F_{\mathbf{h}}\right)\right)^2}$$

and hence:

$$\begin{aligned}
&C_d\left(P^*\right) \\
&\geq \log\left(P^*\right) - \left(\exp\left(H\left(F_{\mathbf{h}}\right)\right) + \epsilon_K + \frac{C\sigma^2}{K\epsilon_K^2} + \frac{\exp^{1/2}(1)\ \sigma}{K^{1/2}\left(C - H\left(F_{\mathbf{h}}\right)\right)}\right) \\
&\geq \log\left(P^*\right) - \left(\exp\left(H\left(F_{\mathbf{h}}\right)\right) + \frac{1}{\log(K)} + \frac{C\sigma^2\log^2(K)}{K} + \frac{\exp^{1/2}(1)\ \sigma}{K^{1/2}\left(C - H\left(F_{\mathbf{h}}\right)\right)}\right)
\end{aligned} \tag{38}$$

■

An illustration is shown in Figure 3.

1) Impact of power delay profile: The impact of the PDP has been touched already in Cor. 2 showing its asymptotic optimality. Similar to the low SNR regime we have an upper bound describing generally the impact of PDP.

Proposition 10: For sufficiently large P^* (and K) the following upper bound holds:

$$C_d\left(P^*\right) \leq \log\left(\frac{P^*}{\mathbb{E}_{\mathbf{h}}\left(\|\mathbf{c}\|_1^{-1}\right)}\right)$$

The bound is independent of the order of the elements of the PDP and concave (thus Schur-concave).

Proof: The proof follows immediately from the geometric mean inequality, i.e. for any $\mathbf{h}$ we have

$$\overline{h} \geq \frac{K}{\|\mathbf{h}\|_1} = \frac{1}{\|\mathbf{c}\|_1}.$$

Taking expectations on both sides in combination with Prop. B.2. in [16] yields the desired result. ■

If L is large with uniform PDP the bound approaches the high SNR AWGN capacity and is thus too optimistic in general.

C. Convergence to the ergodic capacity

Let us now treat the ergodic case. The ergodic capacity is given by [4]:

$$C_e = \mathbb{E}_{\mathbf{h}}\left(\max\left\{\log\left(\xi h_1\right), 0\right\}\right) \quad \text{where } \xi \text{ such that}$$
$$P^* = \mathbb{E}_{\mathbf{h}}\left(\max\left\{\xi - h_1^{-1}, 0\right\}\right)$$

Note that for low SNR the first order term of the DLC is not bounded with respect to L. Hence, the ergodic capacity has the same property which is in accordance with results in [9]. A similar convergence can be shown for the high SNR case.

Corollary 4: Under the assumptions of Lemma 3 the DLC converges to the ergodic capacity as $K \to \infty$.

Proof: We only have to show that C_e scales as $\log(P^*) + H_F$ as $P^* \to \infty$. We can again use a truncation argument. Let $h_1^\epsilon := \max\{h_1, \epsilon\}$. Then, we have for sufficiently large P^*

$$\xi - \mathbb{E}_{\mathbf{h}}\left((h_1^\epsilon)^{-1}\right) = P^*$$

and

$$\begin{aligned} C_e &= \mathbb{E}_{\mathbf{h}}\left(\log\left(\xi h_1^\epsilon\right)\right) \\ &= \mathbb{E}_{\mathbf{h}}\left(\log\left(\left(\mathbb{E}_{\mathbf{h}}\left((h_1^\epsilon)^{-1}\right) + P^*\right) h_1^\epsilon\right)\right) \\ &= \log(P^*) + \mathbb{E}_{\mathbf{h}}\left(\log\left(h_1^\epsilon\right)\right) + \log\left(\mathbb{E}_{\mathbf{h}}\left((h_1^\epsilon)^{-1}\right)(P^*)^{-1} + 1\right) \end{aligned}$$

Now, again by a bounded convergence argument

$$\lim_{\epsilon \downarrow 0} \mathbb{E}_{\mathbf{h}}\left(\log\left(h_1^\epsilon\right)\right) = \mathbb{E}_{\mathbf{h}}\left(\log\left(h_1\right)\right)$$

and

$$\frac{C_e}{\log(P^*) + H_F} \to 1, \quad P^* \to \infty,$$

which completes the proof. ■

We have the following coding implications:

V. OFDM BROADCAST CHANNEL DLC REGION

The delay-limited region is defined as follows:

Definition 2: A rate vector $\mathbf{R}^*$ lies in the DLC region $\mathcal{C}_{\text{DL}}(P^*)$ with sum power constraints P^* constraints if and only if for any fading state $\mathbf{h}$ there is P' solving

$$\min P \quad \text{s.t.} \quad R^* \in \mathcal{C}_{\text{BC}}(\mathbf{h}, P)$$

and

$$\mathbb{E}_{\mathbf{h}}(P') \leq P^*$$

Furthermore, R^* is on the boundary $\mathcal{B}_{\text{DL}}(P^*)$ if and only if

$$\mathbb{E}_{\mathbf{h}}(P') = P^*.$$

To evaluate the OFDM delay-limited region turns out to be very difficult. This is due to the fact, that we have only an *implicit* characterization for the DLC. This means, that we can check for every rate vector $\mathbf{R}$, whether it lies inside the DLC region or not, simply by solving the dual problem for each $\mathbf{h}$. To evaluate the expectation necessary to check the average power condition can be done by Monte-Carlo runs. However, it is very difficult to determine all rate vectors, which can be achieved with a fixed sum power constraint P^*.

Nevertheless and although computationally demanding, the OFDM-DLC region can be calculated up to any desired finite accuracy. To this end we restate the Algorithm 1 from [10] and define

$$n_{m,k} := \log\left\{ e^{\sum\limits_{n>m} R_{\pi_k(n),k}} \left[\frac{\sigma^2}{|h_{\pi_k(m),k}|^2} + \sum_{j=1}^{m-1} \frac{\sigma^2}{|h_{\pi_k(j),k}|^2} \left(e^{R_{\pi_k(j),k}} - 1\right) e^{\sum\limits_{n=j+1}^{m-1} R_{\pi_k(n),k}} \right] \right\}^{-1} \tag{39}$$

$$R_{\pi_k(m),k} = \Big[\log(\mu_m) + n_{m,k}\Big]^+. \tag{40}$$

Then Algorithm 1 yields the minimum sum power necessary to support a set of rates $\mathbf{R}$.

Algorithm 1 Iterative "Rate Water-Filling"

Set $R_{m,k} = 0 \quad \forall m \in \mathcal{M},\ k = 1, ..., K$

while desired accuracy is not reached **do**

  for $m = 1$ to M **do**

    (1) Compute the coefficients $n_{m,k}$ (39) for user m

    (2) Do water-filling with respect to the rates $R_{m,k}$ for user m according to equation (40)

  end for

end while

To evaluate the region $\mathcal{C}_{\text{DL}}(P^*)$, first the single user DLC rates have to be calculated for all m. This is done by the evaluation of (7) for fixed $\mathcal{C}_{\text{DL}}(P^*)$ and bisection, since $\mathcal{C}_{\text{DL}}(P^*)$ is monotone in P^*. Due to the convexity of $\mathcal{C}_{\text{DL}}(P^*)$, any convex combination $\mathbf{R}_{\text{int}}$ must lie inside $\mathcal{C}_{\text{DL}}(P^*)$. On the other hand, the single user rates R_m are a component-wise upper bound for all other rate vectors. Since the necessary power $P(\alpha\mathbf{R}_{\text{int}})$ is monotone in α, simple bisection

can determine the boundary of the region for each angle. For any new points on the boundary, the refinement procedure can be repeated until the desired number of points defining the border is obtained. This procedure is summarized in Algorithm 2. Note, that if the distribution and number of taps is the same for all users, the region is symmetric and can be constructed by mirroring one calculated sector.

Algorithm 2 OFDM DLC Region Algorithm

(1) Determine single user DLC (axis points) by evaluation of single user DLC (see below) and bisection

while desired accuracy not reached **do**

(2) For any two neighboring vectors $\mathbf{R}_1 \in \mathcal{B}_{\mathrm{DL}}(P^*)$ and $\mathbf{R}_2 \in \mathcal{B}_{\mathrm{DL}}(P^*)$ on the boundary calculate interpolated vector $\mathbf{R}_{int} = 1/2\,(\mathbf{R}_1 + \mathbf{R}_2)$

(3) Adjust $\alpha > 1$ by bisection using Alg. 1 such that $\alpha\mathbf{R}_{\mathrm{int}} \in \mathcal{B}_{\mathrm{DL}}(P^*)$

end while

VI. OFDMA ACHIEVABLE DELAY LIMITED RATE REGION

Compared to the single user case the multiuser DLC region is more difficult to analyze. In order to get some insight in this case we derive simple resource allocation schemes based on OFDMA and rate water-filling. To this end, we assume independence of the subcarriers achieved by complex Gaussian distributed path gains and uniform PDP. It is worth noting that this can be imagined as that we take only L independent frequency samples and assume that the other value are approximately equal in the neighborhood of the L subcarriers. Although seeming quite complicated, the following lemma yields a simple lower bound on the OFDM DLC region, implying an OFDMA strategy.

Lemma 4: Let $\mathbf{i} = (i_1 \ldots, , i_K) \in [1, M]^K$ be a multi-index and let the set $\mathcal{K}_m^{\mathbf{i}}$ count the number of user indices in $\mathbf{i}$ equal to m. Let $\mathcal{I}_s \subset [1, M]^K$ be the subset that contains these multi-indices, where all users occur in the multi-index $\mathbf{i}$ at least s times, i.e.

$$\mathcal{I}_s = \left\{ \mathbf{i} : \mathcal{K}_m^{\mathbf{i}} \geq s, m \in \mathcal{M} \right\}.$$

Then the average required power P^* to support any rate vector $\mathbf{R}$ in each fading state is upper

bounded by

$$P^* \leq \sum_{\mathbf{i}\in\mathcal{I}_s}\sum_{m=1}^{M}\sum_{\substack{p=1\\k[p]\in\mathcal{K}_m^{\mathbf{i}}}}^{\left|\mathcal{K}_m^{\mathbf{i}}\right|}\frac{e^{R_{m,k[p]}^{\mathbf{i}}}-1}{M^K\,\zeta_{m,p}^{-1}}+\left(M^K-|\mathcal{I}_s|\right)\sum_{m=1}^{M}\sum_{p=1}^{\lfloor K/M\rfloor}\frac{\left(e^{\bar{R}_{m,k[p]}}-1\right)}{M^K\,\theta_{m,p}^{-1}}$$

where $R_{m,k[p]}^{\mathbf{i}}$ is the solution to

$$\left[\frac{e^{R_{m,k[p]}^{\mathbf{i}}}}{\zeta_{m,p}^{-1}}-\lambda\right]^{-}=0,\quad \sum_{p=1}^{\left|\mathcal{K}_m^{\mathbf{i}}\right|}R_{m,k[p]}^{\mathbf{i}}=KR_m$$

and

$$\zeta_{m,p}=\int_0^{\infty}\frac{1}{x}\,dF_{h'_{k[p]}}(x) \tag{41}$$

with $h'_{k[p]}$ being the p-th order statistic of $\left|\mathcal{K}_m^{\mathbf{i}}\right|$ random variables h' with cumulative density function

$$F_{h'}(x)=\left(1-e^{-x}\right)^M.$$

$\bar{R}_{m,k[p]}$ is the solution to

$$\left[\frac{e^{\bar{R}_{m,k[p]}}}{\theta_{m,p}^{-1}}-\bar{\lambda}\right]^{-}=0,\quad \sum_{p=1}^{\lfloor K/M\rfloor}\bar{R}_{m,k[p]}=KR_m$$

and

$$\theta_{m,p}=\int_0^{\infty}\frac{1}{x}\,dF_{h''_{k[p]}}(x)$$

with $h''_{k[p]}$ being the p-th order statistic of $\left\lfloor\frac{K}{M}\right\rfloor$ random variables h'' with cumulative density function

$$F_{h''}(x)=1-M\int_x^{\infty}\left(1-e^{-x'}\right)^{M-2}\left(e^{x}-e^{-x'}\right)e^{-x'}\,dx'.$$

Proof: The basic idea is to distinguish between two cases: The case, where each user has the best channel at least on s subcarriers and the case where at least one user has on less than s of the K subcarriers the best channel. With the multi-index $\mathbf{i}=(i_1\ldots,,i_K)\in[1,M]^K$ define the event

$$\mathcal{H}_{\mathbf{i}}:=\left\{\omega: h_{i_1,1}(\omega)>h_{l,1}(\omega)|_{l\neq i_1},\ldots,h_{i_K,1}>h_{l,K}(\omega)|_{l\neq i_K}\right\}.$$

Note that by the absolute continuous fading distribution, we have

$$\sum_{\mathbf{i}\in\mathcal{I}}\Pr\left(\mathcal{H}_{\mathbf{i}}\right)=1$$

since the remaining events occur with zero probability. Thus, we can express the average power P^* by

$$P^* = \mathbb{E}_{\mathbf{h}}\left(P'\right) = \frac{1}{M^K}\sum_{\mathbf{i}\in\mathcal{I}_s}\mathbb{E}_{\mathbf{h}}\left(P'\middle|\,\mathcal{H}_{\mathbf{i}}\right) + \frac{1}{M^K}\sum_{\mathbf{i}\notin\mathcal{I}_s}\mathbb{E}_{\mathbf{h}}\left(P'\middle|\,\mathcal{H}_{\mathbf{i}}\right)$$

where $\mathcal{I}_s \subset [1,M]^K$ is the subset that contains the elements where all users occur in the multi-index $\mathbf{i}$ at least s times. Let $\mathcal{K}_m^{\mathbf{i}}$ be the set that counts the number of user indices in $\mathbf{i}$ equal to m. Fixing $\mathbf{i}$ and m and ordering the values according to

$$h_{m,k\left[\left|\mathcal{K}_m^{\mathbf{i}}\right|\right]} \geq \ldots \geq h_{m,k[1]}, \quad k\left[p\right] \in \mathcal{K}_m^{\mathbf{i}}$$

the first term on the right hand side is bounded by

$$\sum_{\mathbf{i}\in\mathcal{I}_s}\mathbb{E}_{\mathbf{h}}\left(P'\middle|\,\mathcal{H}_{\mathbf{i}}\right) \leq \sum_{\mathbf{i}\in\mathcal{I}_s}\sum_{m=1}^{M}\sum_{\substack{p=1\\ k[p]\in\mathcal{K}_m^{\mathbf{i}}}}^{\left|\mathcal{K}_m^{\mathbf{i}}\right|}\mathbb{E}_{\mathbf{h}}\left(\frac{e^{R^{\mathbf{i}}_{m,k[p]}}-1}{M^K\,h_{m,k[p]}}\middle|\,\mathcal{H}_{\mathbf{i}}\right) \tag{42}$$

with $R^{\mathbf{i}}_{m,k[p]}$ such that the required rates are supported: Since the expectation on the RHS is independent of the actual referred subindex in the multi-index $\mathbf{i}$ and depends only on the number of emerging entries of user m in $\mathbf{i}$ counted by the set $\mathcal{K}_m^{\mathbf{i}}$ we can replace the RHS and rewrite the inequality in (42) as

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{I}_s}\mathbb{E}_{\mathbf{h}}\left(P'\middle|\,\mathcal{H}_{\mathbf{i}}\right) &\leq \sum_{\mathbf{i}\in\mathcal{I}_s}\sum_{m=1}^{M}\sum_{\substack{p=1\\ k[p]\in\mathcal{K}_m^{\mathbf{i}}}}^{\left|\mathcal{K}_m^{\mathbf{i}}\right|}\frac{e^{R^{\mathbf{i}}_{m,k[p]}}-1}{M^K}\mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_{m,k[p]}}\right)\\ &\leq \sum_{\mathbf{i}\in\mathcal{I}_s}\sum_{m=1}^{M}\sum_{p=1}^{\left|\mathcal{K}_m^{\mathbf{i}}\right|}\frac{e^{R^{\mathbf{i}}_{m,k[p]}}-1}{M^K\,\zeta_{m,p}^{-1}}\end{aligned}$$

where $\zeta_{m,p}$ is the expectation of the p-th inverse channel coefficient. The distribution of the p-th order of the best channels $h_k^{\max}$ for some k is independent of k and given in equation (9). The cdf and pdf $f(x)$ of $h_k^{\max}$ in turn can also be derived by the order statistic from (9) and be expressed as

$$\begin{aligned}F^{(c)}(x) &= \frac{\Pr\left(h_{1,1} > x, h_{1,1} > h_{l,1,l\neq 1}\right)}{\Pr\left(h_{1,1} > h_{l,1,l\neq 1}\right)}\\ &= M\int_x^{\infty}\left(1-e^{-x'}\right)^{M-1}e^{-x'}\,dx'.\end{aligned} \tag{43}$$

and

$$f(x) = -\frac{dF^c(x)}{dx} = M\left(1-e^{-x}\right)^{M-1}e^{-x} \tag{44}$$

Substituting (43) and (44) into (9) yields the desired densities.

For the second term we need to carry out a different strategy. For all cases represented in the complementary set $\bar{\mathcal{I}}_s$ at least one user occurs in the multi-index $\mathbf{i}$ less than s times and hence has the best channel on less than s subcarriers. For the case $s = 1$, the strategy of the previous term can not even guarantee his delay limited rate requirement. Alternatively, we simply divide the set of subcarriers in M sub-bands where to each user $\lfloor K/M \rfloor$ subcarriers are allocated and do rate water-filling as done for the first term and take the best out of this set. Hence the second term is upper bounded by

$$\begin{aligned}\sum_{\mathbf{i}\in\bar{\mathcal{I}}_s} \mathbb{E}_{\mathbf{h}}\left(P' \middle| \mathcal{H}_{\mathbf{i}}\right) &\leq \sum_{\mathbf{i}\in\bar{\mathcal{I}}_s} \mathbb{E}_{\mathbf{h}}\left(\sum_{m=1}^{M}\sum_{p=1}^{\lfloor K/M\rfloor} \frac{e^{\bar{R}_{m,k[p]}}-1}{h_{m,k[p]}} \middle| \mathcal{H}_{\mathbf{i}}\right) \\ &\leq \sum_{\mathbf{i}\in\bar{\mathcal{I}}_s}\sum_{m=1}^{M}\sum_{p=1}^{\lfloor K/M\rfloor}\left(e^{\bar{R}_{m,k[p]}}-1\right)\mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_{m,k[p]}}\middle| \mathcal{H}_{\mathbf{i}}^{m}\right)\end{aligned} \tag{45}$$

where the set $\mathcal{H}_{\mathbf{i}}^{m}$ is defined as

$$\mathcal{H}_{\mathbf{i}}^{m} := \left\{\omega : h_{i_{k_m^l},k_m^l}(\omega) > h_{m,k_m^l}(\omega)\Big|_{m\neq i_{k_m^l}}, \ldots, h_{i_{k_m^u},k_m^u}(\omega) > h_{m,k_m^u}(\omega)\Big|_{m\neq i_{k_m^u}}\right\}. \tag{46}$$

Note, that the second inequality stems from the fact, that the expectation is conditioned on the set $\mathcal{H}_{\mathbf{i}}^{m}$, assuming that user m has not the best channel on any of his subcarriers.

Since all subcarriers are independent, we define for each subcarrier the following conditioned probability and get after some manipulations

$$\begin{aligned}F_{h''}(x) &= 1 - \Pr\left(h_{m,k} > x \middle| h_{n,k} > h_{m,k}, n\neq m\right) \\ &= 1 - \frac{\Pr\left(h_{1,1} > x, h_{2,1} > h_{l,1,l\neq 2}\right)}{\Pr\left(h_{2,1} > h_{l,1,l\neq 2}\right)} \\ &= 1 - M\int_x^{\infty}\left(1-e^{-x'}\right)^{M-2}\left(e^{x}-e^{-x'}\right)f\left(x'\right)\,dx'.\end{aligned} \tag{47}$$

Thus, with (9) we can express the condition expectation as

$$\mathbb{E}_{\mathbf{h}}\left(\frac{1}{h_{m,k[p]}}\middle| \mathcal{H}_{\mathbf{i}}^{m}\right) = \int_0^{\infty}\frac{1}{x}\,dF_{h''_{k[p]}}(x)$$

leading to (41). Since the addends do not depend on the index $\mathbf{i}$, the first sum in (45) can be substituted by the factor $\left|\bar{\mathcal{I}}_s\right| = M^K - \left|\mathcal{I}_s\right|$ leading to

$$\sum_{\mathbf{i}\in\bar{\mathcal{I}}_s}\mathbb{E}_{\mathbf{h}}\left(P'\middle|\mathcal{H}_{\mathbf{i}}\right) \leq \left(M^K - \left|\mathcal{I}_s\right|\right)\sum_{m=1}^{M}\sum_{p=1}^{\lfloor K/M\rfloor}\frac{\left(e^{\bar{R}_{m,k[p]}}-1\right)}{M^K\,\theta_{m,p}^{-1}}.$$

This concludes the proof. ■

Note, that for the case $M = 2$, the expression for the cdf in (47) simplifies to $F_{h''}(x) = 1 - e^{-2x}$. Instead of partitioning the subcarriers equally among the users, it is possible to share them in any other relation. Then the second sum of the second term in (4) is not has not $\lfloor K/M \rfloor$ addends but K_m addends for each user with $\sum_{m=1}^{M} K_m = K$. So it is especially reasonable to share the subcarriers proportional to the users rate requirements such that $K_m = R_m / \sum_{m=1}^{M} R_m$.

The bounds from the previous section are illustrated in Fig. 8 and Fig. 7. The bounds are shown for different values of s, changing the relation between term 1 and term 2 in (4). Fig. 7 depicts the low SNR case. It can be seen that the lower bound nearly achieves the entire region. This is due to the fact that in the low SNR regime only the best subcarrier is used. This is realized with rate water-filling, even if not perfect but only ordinal information, e.g. the ranking of the subcarriers, is present. The remaining gap stems from the second term and the fact, that users can ”collide”, i.e. have a common best subcarrier.

In contrast, in Fig. 8 the high SNR scenario is presented. The bound improves as s is increased up to $s = 4$. From $s = 5$ on, the bound degrades once again. This is since it is not optimal to support the entire rate only on one subcarrier, even if a user has only one best subcarrier. Thus, the bound improves as s is increased. Note, that for the case that both users have similar rate requirements, i.e. the *sum DLC case*, the bound achieves a major part of the DLC and outperforms the time-sharing strategy. The remaining gap on the axes is much bigger. However, the gap on the axes can be reduced by sharing the subcarriers proportional to the users rate requirements and using only the second term (thus making the conditioning of the pdf needless). This is illustrated with the curve called *prorated FDMA*. The discontinuity stems from the switching of the subcarrier allocation, since this is a discrete procedure. The dashed blue curve indicates an *achievable OFDMA DLC region*, which is given in Algorithm 3: For any rate vector $\mathbf{R}$ and any channel realization $\mathbf{h}$ the sum power minimization algorithm is usd to calculate the optimal resource allocation. The resulting Lagrangian multipliers $\boldsymbol{\mu}$ are taken to allocate the subcarriers according to the maximum weighted channel rule $m_k = \arg\max_{m \in \mathcal{M}} \mu_m h_{m,k}$. Once, the subcarrier allocation is done, the optimal resource allocation is obtained by water-filling such that the rate requirements are met.

This scheme requires perfect CSI but is computationally still relatively simple due to the iterative water-filling principle. It can be seen from Fig. 8 that the algorithm yields good results

Algorithm 3 OFDMA DLC Algorithm

(1) for given rate vector $\mathbf{R}$ and channel realization $\mathbf{h}$ solve the minimum sum power problem with Alg 1

(2) assign subcarriers according to $m_k = \arg\max_{m\in\mathcal{M}} \mu_m h_{m,k}$

for $m = 1$ to M **do**

(3) do water-filling with respect to the rates $R_{m,k}$ for user m such that rate requirement R_m is met

end for

where any other FDMA approach has to compete with.

VII. Conclusions

We studied the delay limited capacity of OFDM systems. We have shown that explicite expressions can be found in the low and high SNR regime even for the challenging correlation structure of OFDM. Even though we presented our results in the context of OFDM they are not restricted to this class but apply to other channels such as MIMO as well. On the other hand, still a basic open problem is the complete characterization of the DLC for all SNR and arbitrary power delay profile. Here, we were not able to give universal bounds and it is an interesting problem to show that the dependence is in general so-called Schur-concave which implies that a uniform profile maximizes the DLC in all cases. Furthermore, we analyzed the OFDM BC DLC region and derived lower bounds based on rate water-filling. In the low SNR regime and concerning DLC throughput, these bounds perform very well. To approach the DLC close to the axes in the high SNR regime, a prorated strategy has to be used. All bounds merely use order statistics and involve only ordinal – and thus partial – channel knowledge, which suggests savings for the design of future feedback protocols. Moreover, an additional FDMA strategy using full channel state information is proposed, performing very well over the entire region.

Appendix

A. *Proof of Theorem 1*

By Theorem 3 in [13] we have to check that:

- $\mathbb{E}_{\mathbf{h}}\left(\exp\left[j\omega\Re e\left(\sum_{l=1}^{L}\tilde{c}_l\, e^{-jl\theta_k}\right)\right]\right) = e^{-\frac{\omega^2}{4}+\sum_{i=3}^{5}a_i\omega^i+O\left(\omega^6\right)}$ holds for any $\theta_k := 2\pi k/K, k=0,\ldots,K-1$, and for any real number ω in the non-empty interval $[-d,d]$ for some $d>0$, and furthermore
- $\frac{1}{K^2}\sum_{k_1=1}^{K}\sum_{k_2=1,k_2\neq k_1}^{K}\mathbb{E}_{\mathbf{h}}\left(\exp\left[j\Re e\left(\sum_{l=1}^{L}\tilde{c}_l\left(\omega_1 e^{-jl\theta_{k_1-1}}+\omega_2 e^{-jl\theta_{k_2-1}}\right)\right)\right]\right)$ $= e^{-\frac{\omega_1^2}{4}-\frac{\omega_2^2}{4}+\sum_{i=3}^{5}a_i(|\omega_1|+|\omega_2|)^i+O\left((|\omega_1|+|\omega_2|)^6\right)}$ holds for all real numbers ω_1,ω_2 in $[-d,d]^2$.

We show only the more complicated second condition. The first condition can be easily deduced by our assumptions on the distributions and observing that the subcarriers' real and imaginary parts are independent. We have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}}\left(\exp\left[j\Re e\left(\sum_{l=1}^{L}\tilde{c}_l\left(\omega_1 e^{-jl\theta_{k_1}}+\omega_2 e^{-jl\theta_{k_2}}\right)\right)\right]\right)\\&= e^{-\frac{1}{4L}\sum_{l=1}^{L}\left(\omega_1\cos\left(l\theta_{k_1}\right)+\omega_2\cos\left(l\theta_{k_2}\right)\right)^2+\left(\omega_1\sin\left(l\theta_{k_1}\right)+\omega_2\sin\left(l\theta_{k_2}\right)\right)^2}e^{O\left((|\omega_1|+|\omega_2|)^3\right)}.\end{aligned}$$

By analytic expansion of the first factor and observing that

$$\begin{aligned}&\frac{1}{4L}\sum_{k_1=0}^{K-1}\sum_{k_2=0,k_2\neq k_1}^{K-1}\sum_{l=1}^{L}\left(\omega_1\cos\left(l\theta_{k_1}\right)+\omega_2\cos\left(l\theta_{k_2}\right)\right)^2+\left(\omega_1\sin\left(l\theta_{k_1}\right)+\omega_2\sin\left(l\theta_{k_2}\right)\right)^2\\&=\frac{\omega_1^2}{4}+\frac{\omega_2^2}{4}+2\omega_1\omega_2\sum_{k_1=0}^{K-1}\sum_{k_2=0,k_2\neq k_1}^{K-1}\sum_{l=1}^{L}\left(\cos\left(l\theta_{k_1}\right)\cos\left(l\theta_{k_2}\right)+\sin\left(l\theta_{k_1}\right)\sin\left(l\theta_{k_2}\right)\right)\\&=\frac{\omega_1^2}{4}+\frac{\omega_2^2}{4}\end{aligned}$$

where the last step is due to the standard trigonometric relation

$$\sum_{k=0}^{K-1}\cos\left(l\theta_k\right)=\begin{cases}K & l=0\\ 0 & l\neq 0, l<K\end{cases}$$

we have finally

$$\begin{aligned}&e^{\left(-\frac{1}{4L}\sum_{l=1}^{L}\left(\omega_1\cos\left(l\theta_{k_1}\right)+\omega_2\cos\left(l\theta_{k_2}\right)\right)^2+\left(\omega_1\sin\left(l\theta_{k_1}\right)+\omega_2\sin\left(l\theta_{k_2}\right)\right)^2+O\left((|\omega_1|+|\omega_2|)^3\right)\right)}\\&= e^{-\frac{\omega_1^2}{4}-\frac{\omega_2^2}{4}+\sum_{i=3}^{5}a_i(|\omega_1|+|\omega_2|)^i+O\left((|\omega_1|+|\omega_2|)^6\right)}\end{aligned}$$

which is the desired result. The proof that non-uniform PDP can not improve this bound follows from union bound and is omitted [13].

B. Proof of Lemma 2

The order of the lower and upper bound was already derived in [13]. Due to the correlation structure imposed by the FFT and Rayleigh fading with uniform PDP the channel gains $h_{k_1+(a-1)k_2}, k_2 \in \{1, \dots, L\}$, are independent for any $k_1 \in \{1, \dots, a\}$ for some $a \in \mathbb{N}$ where $a = K/L$. Since the maximum $\left\| \left(h_{k_1}, h_{k_1+a}, \dots, h_{k_1+a(L-1)} \right) \right\|_\infty$ is below or equal x if h_∞ is below or equal x but not conversely, it follows the lower bound [13]

$$\Pr \left(h_\infty \le x \right) \le e^{-L^\varepsilon}$$

which is the desired lower bound if we set $\varepsilon = \gamma \log \left[\log \left(L \right) \right] / \log \left(L \right)$ (in fact, obviously, it is even stronger and can be to strengthened to fall off with L^{-1} instead of order $\log^{-1} \left(L \right)$). The upper bound is obtained by observing that for any $a \ge 1$ we have by the FFT structure

$$\Pr \left(h_\infty > x \right) \le a \, \left[1 - \left(1 - e^{-x} \right)^L \right].$$

Setting this time $x = (1 + \varepsilon) \log \left(L \right)$ for some $\varepsilon > 0$ yields

$$\begin{aligned}
\Pr \left(h_\infty > (1 + \varepsilon) \log \left(L \right) \right) &\le a \, \left[1 - \left(1 - \frac{1}{L^{(1+\varepsilon)}} \right)^L \right] \\
&= a \, \left(1 - \exp \left[L \log \left(1 - \frac{1}{L^{(1+\varepsilon)}} \right) \right] \right) \\
&\le a \, \left(1 - \exp \left[- \frac{L^{-\varepsilon}}{1 - L^{-(1+\varepsilon)}} \right] \right) \\
&\le \frac{a \, L^{-\varepsilon}}{1 - L^{-(1+\varepsilon)}} \\
&\le \frac{aL}{L - L^{-\varepsilon}} L^{-\varepsilon}
\end{aligned}$$

using $\log \left(1 - x \right) \ge - \frac{x}{1-x}$ and $e^{-x} \ge 1 - x$. Hence, if we set $\varepsilon = \frac{\gamma \log[\log(L)]}{\log(L)}$ we have

$$\Pr \left(h_\infty > \log \left(L \right) + \log \left[\log \left(L \right) \right] \right) \le \frac{\kappa}{\log^\gamma \left(L \right)}.$$

Combining this with the stronger lower bound yields the result.

C. Proof of Proposition 7

We can frequently use the following well-known inequality: suppose that $f_1, \dots, f_K$ are functions defined on a domain Ω equipped with some measure F with $f_k \in \mathcal{L}^{p_k} \left(\Omega \right)$ and

$\sum_{i=1}^{K} p_k^{-1} = 1$ then [17]:

$$\left| \int_{\Omega} \prod_{k=1}^{K} f_k\left(x\right)\, dF\left(x\right) \right| \leq \prod_{k=1}^{K} \left(\int_{\mathbb{R}_+} f^{p_k}\left(x\right)\, dF\left(x\right) \right)^{1/p_k} \tag{48}$$

This inequality is tailored for the situation at hand: suppose that for some $a, b > 0$

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) &= \int_{\mathbb{R}_+^K} \prod_{k=1}^{K} h^{-\frac{1}{K}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right) \\
&= \int_{\mathbb{R}_+^K} \prod_{k_1=1}^{a} \prod_{k_2=1}^{b} h_{k_1+(k_2-1)a}^{-\frac{1}{K}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right)
\end{aligned}$$

which yields by application of (48):

$$\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) \leq \prod_{k_1=1}^{a} \left[\int_{\mathbb{R}_+^K} \prod_{k_2=1}^{b} h_{k_1+(k_2-1)a}^{-\frac{1}{b}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right) \right]^{\frac{1}{a}} \tag{49}$$

The inner term on the RHS of (49) generally means multidimensional integration with usually dependent random variables which cannot be directly carried out. Hence, we resort to some bounding techniques and have to show that an upper bound holds for the inner integral independent of k_1.

In order to obtain an upper bound on the inner term on the RHS of (49) choose some $\epsilon > 0$ and subdivide the integration domain in parts where in each dimension the range of integration is either in the interval $[0, \epsilon]$ or outside this interval. For those dimensions that are within this interval we bound the corresponding marginal density by the constant c_s and calculate the remaining integral while for those dimensions that are outside the interval we simply set the values of the integrand to ϵ. Suppose that l dimensions are within the interval. Since it does not matter what particular dimensions are chosen for this decomposition we have l out of b possibilities that can be equally treated. Hence, we can write

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right) &\leq \int_{\mathbb{R}_+^K} \prod_{k=1}^{b} h_{k_1+(k_2-1)a}^{-\frac{1}{b}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right) \\
&\leq c_s \sum_{l=0}^{b-1} \binom{b}{l} \prod_{k=1}^{l} \epsilon^{-\frac{1}{b}} \int_{[0,\epsilon]^{b-l}} \prod_{k=1}^{b-l} h_{k_1+(k_2-1)a}^{-\frac{1}{b}}\, dh + \frac{1}{\epsilon} \\
&= c_s \sum_{l=0}^{b} \binom{b}{l} \left(\frac{b}{b-1} \right)^{b-l} \epsilon^{b-l-1} + \frac{1-c_s}{\epsilon},
\end{aligned}$$

and since

$$\sum_{l=0}^{b}\binom{b}{l}(1-\epsilon)^{l}\epsilon^{b-l}=1$$

for any $0\le\epsilon\le 1$ it follows

$$\mathbb{E}_{\mathbf{h}}\left(\overline{h}\right)\le c_s b\left(\frac{b}{b-1}\right)^{b}+(1-c_s)\,b$$

where $\epsilon=1/b<1$.

D. Proof of Proposition 8

We can nicely use Hölder's inequality. Swapping expectation and product operator we have by partial integration

$$\begin{aligned}\int_{\mathbb{R}_+}h^{-\frac{1}{b}}\,f(h)\;dh&=\underbrace{h^{-\frac{1}{b}+1}f(h)\Big|_0^{\infty}}_{=0}-\frac{b}{b-1}\int_{\mathbb{R}_+}h^{-\frac{1}{b}+1}\,f'(h)\;dh\\&=-\frac{b}{b-1}\int_{\mathbb{R}_+}h^{-\frac{1}{b}+1}\,f'(h)\;dh\end{aligned}$$

since f finite everywhere. Define $\mathbb{R}_+^{\circ}:=\{h:f'(h)<0\}$ and let $0\le a_i^-<b_i^-\le\infty, i\in\mathbb{N}$, be the interval boundaries of h where $f'(h)\le 0$ as well as $0\le a_i^+<b_i^+\le\infty, i\in\mathbb{N}$, those where $f'(h)\ge 0$. Representing $(-f')$ as $(-f')=(-f')^{\frac{b-1}{b}}\circ(-f')^{\frac{1}{b}}$ in $\mathbb{R}_+^{\circ}$ yields

$$-\frac{b}{b-1}\int_{\mathbb{R}_+^{\circ}}h^{-\frac{1}{b}+1}\,f'(h)\;dh=\frac{b}{b-1}\int_{\mathbb{R}_+^{\circ}}h^{\frac{b-1}{b}}\left(-f'(h)\right)^{\frac{b-1}{b}}\left(-f'(h)\right)^{\frac{1}{b}}(h)\;dh$$

Setting $p=b/(b-1)$ and $q=b$

$$\begin{aligned}\int_{\mathbb{R}_+^{\circ}}h^{-\frac{1}{b}}\,\left(-f'(h)\right)\,dh&\le\left(\int_{\mathbb{R}_+^{\circ}}\left(-f'(h)\right)\,dh\right)^{\frac{1}{b}}\left(\int_{\mathbb{R}_+^{\circ}}h\left(-\,f'(h)\right)\,dh\right)^{\frac{b-1}{b}}\\&\le\left(\sum_{i\ge1}f\left(a_i^-\right)-f\left(b_i^-\right)\right)^{\frac{1}{b}}\left(\int_{\mathbb{R}_+^{\circ}}h\,\left(-f'(h)\right)\,dh\right)^{\frac{b-1}{b}}\end{aligned}$$

Again by partial integration for the last term

$$-\int_{\mathbb{R}_+^{\circ}}h\,f'(h)\;dh=-\,hf(h)|_0^{\infty}+\int_{\mathbb{R}_+^{\circ}}f(h)\;dh$$

$$\le 1$$

and since

$$-\frac{b}{b-1}\int_{\left(\mathbb{R}_{+}^{\circ}\right)^{C}} h^{-\frac{1}{b}+1}\, f'\left(h\right)\, dh \leq -\frac{b}{b-1}\sum_{i\geq 1} a_i^{\frac{b-1}{b}}\int_{\left[a_i^{+},b_i^{+}\right]} f'\left(h\right)\, dh$$
$$= -\frac{b}{b-1}\sum_{i\geq 1} a_i^{\frac{b-1}{b}}\left(f\left(b_i^{+}\right)-f\left(a_i^{+}\right)\right)$$

we have finally

$$\left(\int_{\mathbb{R}_{+}} h^{-\frac{1}{b}}\, f\left(h\right)\, dh\right)^{b}$$
$$\leq \left(\frac{b}{b-1}\right)^{b}\left[\left(\sum_{i\geq 1} f\left(a_i^{-}\right)-f\left(b_i^{-}\right)\right)^{1/b}-\sum_{i\geq 1} a_i^{\frac{b-1}{b}}\left(f\left(b_i^{+}\right)-f\left(a_i^{+}\right)\right)\right]^{b}$$

which concludes the proof.

E. Proof of Proposition 9

Suppose K, L to be even integers. We can apply inequality (49) with $a = K/2$ and $b = 2$:

$$\int_{\mathbb{R}_{+}^{K}}\prod_{k=1}^{K} h_k^{-\frac{1}{K}}\, dF_{\mathbf{h}}\left(\mathbf{h}\right) \leq \prod_{l=1}^{K/2}\left(\int_{\mathbb{R}_{+}^{K}} h_l^{-1/2} h_{l+K/2}^{-1/2}\, dF_{\mathbf{h}}\left(\mathbf{h}\right)\right)^{2/K}$$

Since $h_k = |\tilde{\mathbf{c}}^T\boldsymbol{\zeta}_k|^2$ with $\boldsymbol{\zeta}_k := [1, e^{-2\pi j(k-1)/K}, ..., e^{-2\pi j(k-1)(K-1)/K}]^T$ we can write for the inner integral

$$\int_{\mathbb{R}_{+}^{K}} h_l^{-1/2} h_{l+K/2}^{-1/2}\, dF_{\mathbf{h}}\left(\mathbf{h}\right) = \iint_{\mathbb{R}^{2L}}\prod_{k=1}^{2}\left|\tilde{\mathbf{c}}^T\boldsymbol{\zeta}_{(k-1)K/2+l}\right|^{-1}\prod_{k=1}^{L} f_{\tilde{c}_k^r}^{r}\left(\tilde{c}_k^r\right) f_{\tilde{c}_k^i}\left(\tilde{c}_k^i\right)\, d\tilde{\mathbf{c}}^r d\tilde{\mathbf{c}}^i \tag{50}$$

where f^r, f^i are the bounded densities of real and imaginary parts of the complex path gains and $(\cdot)^{r,i}$ is a shorthand notation for real and imaginary part operators. Let $l \in [1, K/2]$ be arbitrary but fixed. The following change of coordinates is based on the observation that for K, L even $\boldsymbol{\zeta}_l, \boldsymbol{\zeta}_{l+K/2}$ are orthogonal in $\mathbb{C}^L, \forall l$. For $l = 1$ this is obvious since the first and $K/2$-th vector consist of an even number of binary ± 1's only. For $l > 1$ the same follows from the fact that two orthogonal vectors remain orthogonal if the are both multiplied by the same complex phase factors.

Hence, there exist $\boldsymbol{\zeta}_i^z, i = 2, \ldots L-2$, that can be chosen to span the basis of the orthogonal complement. By change of coordinates $\tilde{\mathbf{c}} \rightarrow \left(\tilde{\mathbf{h}}^z\right), \tilde{h}_1^z = \tilde{\mathbf{c}}^T\boldsymbol{\zeta}_l, \tilde{h}_2^z = \tilde{\mathbf{c}}^T\boldsymbol{\zeta}_{K/2+l}, \tilde{h}_i^z = \tilde{\mathbf{c}}^T\boldsymbol{\zeta}_i^z, i = 2, \ldots L-2$, where $\boldsymbol{\zeta}_l, \boldsymbol{\zeta}_{l+K/2}, \boldsymbol{\zeta}_i^z, \forall i$ is an orthogonal transformation and the Jacobian equals

$1/L^L$ and, further, $\tilde{h}_k^z \to (|\tilde{h}_k^z|^2, \varphi_k^z) = (R_k^z, \varphi_k^z)\,, k=1,\ldots L$, of which the Jacobian is $1/2\ \forall k$ we obtain by assuming $f_{\tilde{c}_k^r}(x) \le v_k^{1/2} e^{-\alpha|x|^2}$ and $f_{\tilde{c}_k^i}(x) \le v_k^{1/2} e^{-\alpha|x|^2}, \forall k$:

$$
\begin{aligned}
\text{RHS of (50)} \le \frac{\pi^L}{L^L} \int_{\mathbb{R}_+^L} (R_1^z)^{-\frac{1}{2}} (R_2^z)^{-\frac{1}{2}} \quad \ldots \\
\prod_{k=1}^{L} v_k^{1/2} e^{-\alpha\left|\left(\frac{1}{L}\boldsymbol{\zeta}^H_{(k-1)K/L+l}\tilde{\mathbf{h}}^z(\mathbf{R}^z,\boldsymbol{\phi}^z)\right)^r\right|^2} v_k^{1/2} e^{-\alpha\left(\frac{1}{L}\boldsymbol{\zeta}^H_{(k-1)K/L+l}\tilde{\mathbf{h}}^z((\mathbf{R}^z,\boldsymbol{\phi}^z))\right)^i} d\boldsymbol{\phi}^z d\mathbf{R}^z \\
= \frac{\pi^L \prod_{k=1}^L v_k}{L^L} \prod_{k=1}^{2} \int_{\mathbb{R}_+} (R_k^z)^{-\frac{1}{2}}\ e^{-\frac{\alpha R_k^z}{L}}\, dR^z \prod_{k=3}^{L} \int_{\mathbb{R}_+} e^{-\frac{\alpha R_k^z}{L}}\, dR^z \\
\le \frac{\pi^L \prod_{k=1}^L v_k}{L^L} \left(\frac{2}{2-1}\right)^2 \left(\frac{L}{\alpha}\right) \left(\frac{L}{\alpha}\right)^{L-2} \qquad \text{(apply Prop. 8)} \\
= \frac{4\pi^L \prod_{k=1}^L v_k}{L^L} \left(\frac{L}{\alpha}\right)^{L-1}
\end{aligned}
$$

Remark 4: For many fading distributions the claim $f_{\tilde{c}_k^{r,i}}(x) \le v_k^{1/2} e^{-\alpha|x|^2}$ might be too restrictive and shall be replaced by $f_{\tilde{c}_k^{r,i}}(x) \le \max\left\{c^{(\max)}, v_k^{1/2} e^{-\alpha|x|^2}\right\}$ where $c^{(\max)} > 0$ is some global constant. The latter inequality, however, does not separate over $\mathbb{R}^L$ as required in the proof here. In this situation, we can obtain better bounds for specific cases by combining the techniques of Prop. 7 - Prop. 9, e.g. by splitting up the integration domain similar to Prop. 7.

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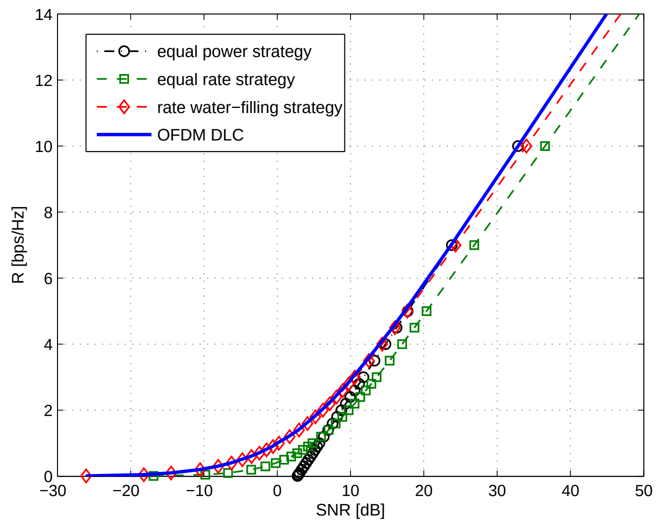

Fig. 1. OFDM DLC and lower bounds for L=K=16.

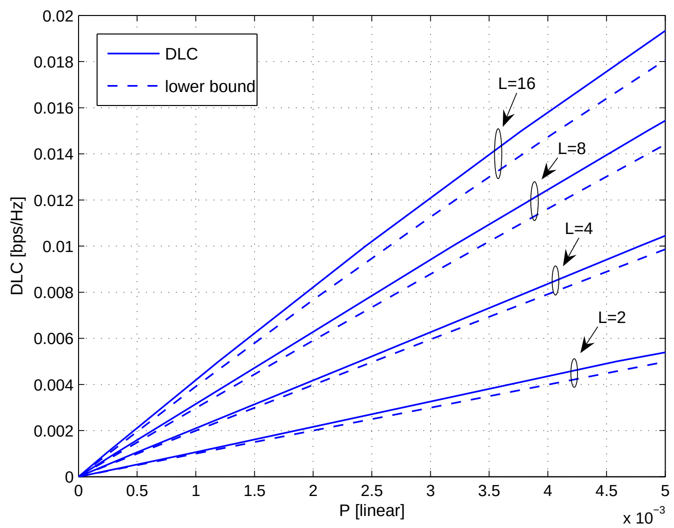

Fig. 2. Scaling in the low SNR region: The dashed lines indicate the scaling at low SNR given by $S_0 = 1/K \log_2(1 + KP^* \log(K))$

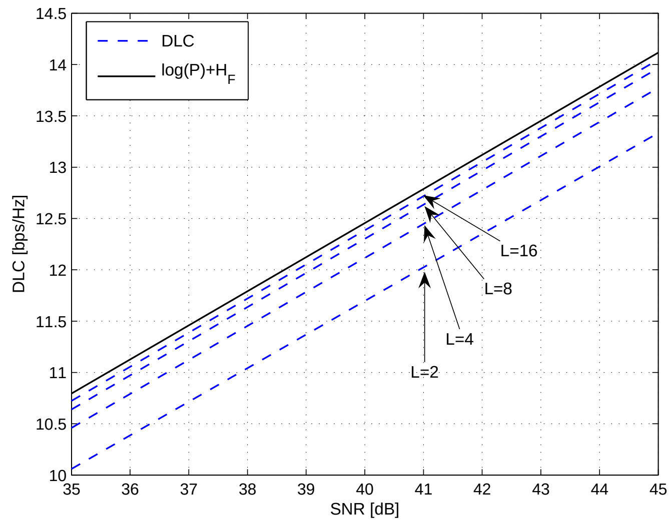

Fig. 3. Scaling in the high SNR region: The black line indicates the scaling at high SNR given by $S = \log(P^*) + H_F$ (for Rayleigh fading). The dashed lines give the OFDM DLC for $L = K = 2, 4, 8, 16$.

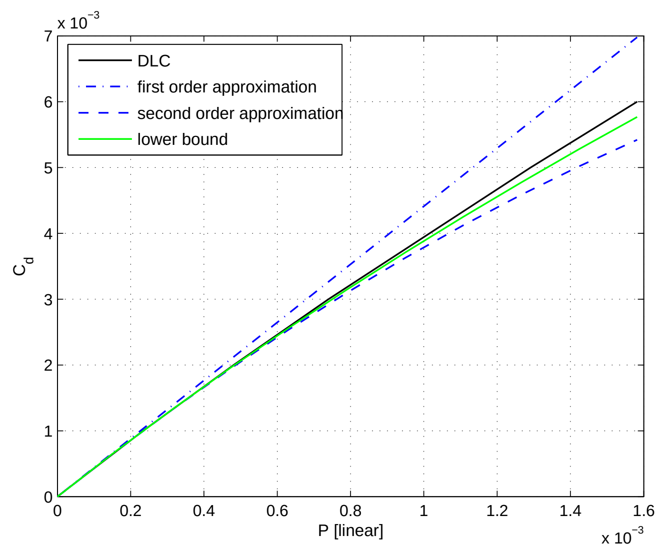

Fig. 4. Comparison of the DLC with the first and second order approximations and the lower bound from Fig. 2. Channel with 64 taps delay spread and system with 64 subcarriers

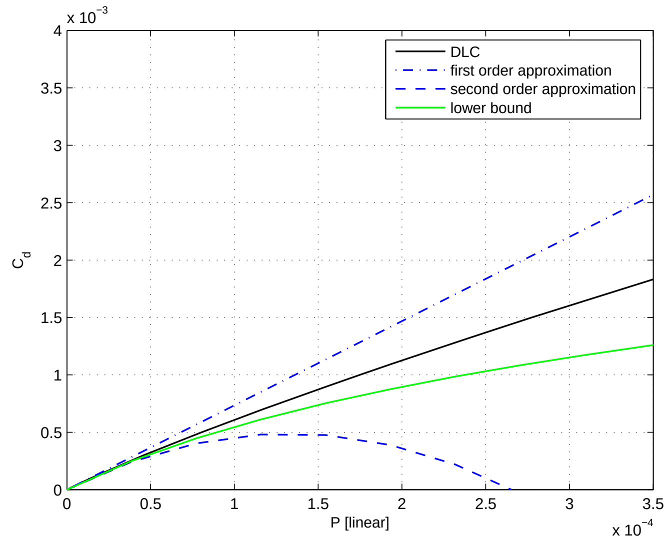

Fig. 5. Comparison of the DLC with the first and second order approximations and the lower bound from Fig. 2. Channel with 1024 taps delay spread and system with 1024 subcarriers

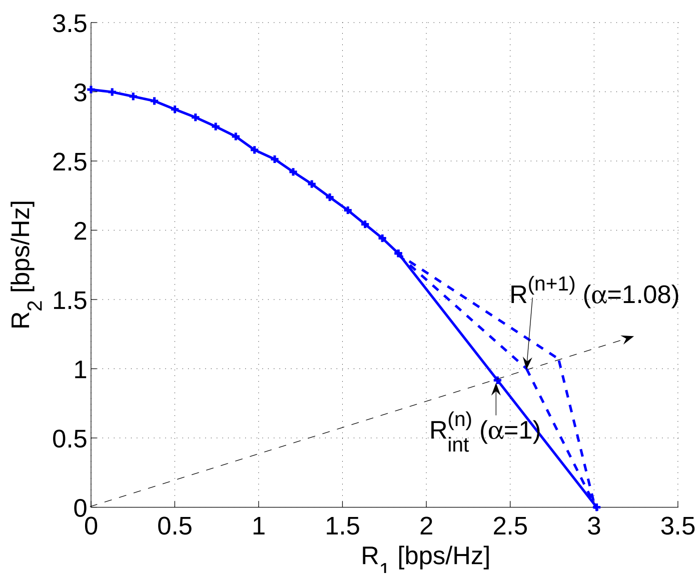

Fig. 6. Example for an iteration of the described algorithm to calculate the OFDM DLC region. OFDM MAC DLC region for 2 users with 7 i.i.d taps each and 16 subcarriers at $10dB$

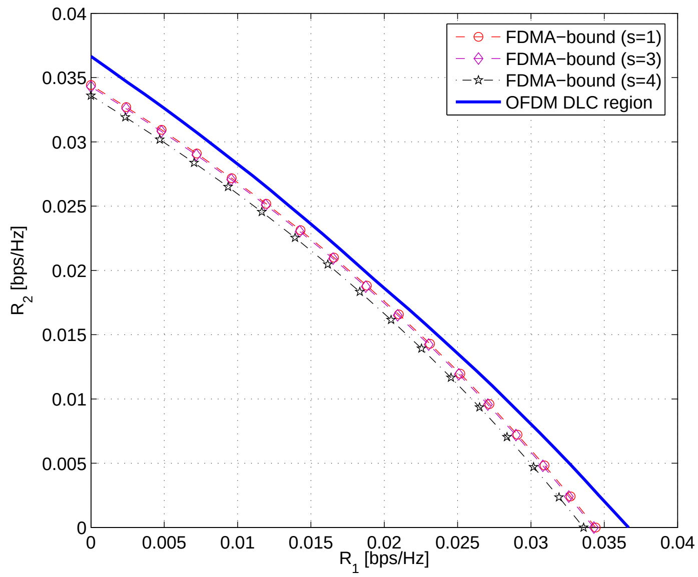

Fig. 7. OFDM DLC region for 16 subcarriers at -20 dB. The lower bound is shown for different values of the parameter s. The lower bound degrades for increasing values of s.

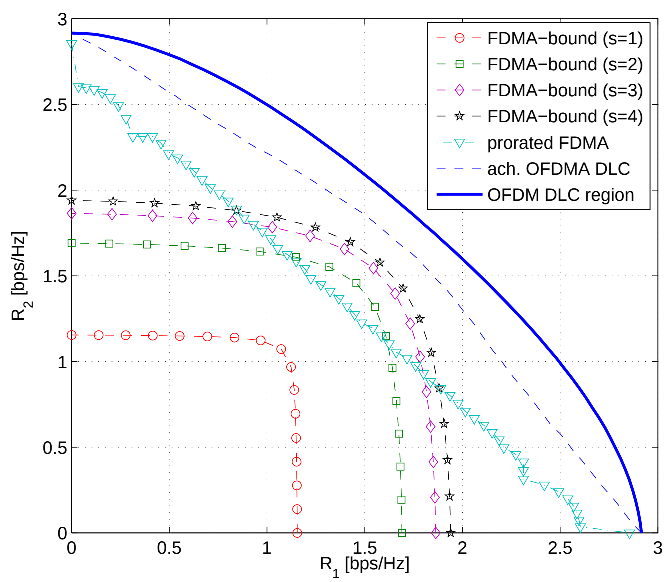

Fig. 8. OFDM DLC region for 16 subcarriers at 10 dB. The lower bound is shown for values of the parameter $s = 1, ..., 4$. Further, the prorated FDMA region and an achievable OFDMA DLC region are depicted.